Awesome Group Theory Worksheet!: Sylow Theorems & Group Actions

I. Cauchy’s Theorem

This is a fun one because we are going to build \(X\) out of \(G\) but use a different group to act on \(X\).

Suppose that \(G\) is a group Let \(X\) = the set of all \(p\)-tuples of elements of \(G\) with the property that product of the entries is the identity.

<table>
<thead>
<tr>
<th>An Example To Help Visualize (X):</th>
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<tbody>
<tr>
<td>Let (G = D_6)</td>
</tr>
<tr>
<td>A. List the elements of (X) if (p = 2)</td>
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<tr>
<td>B. What is (</td>
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Now consider the subgroup \(H\) of \(S_p\) (permutations on a set of \(p\) elements) generated by the \(p\)-cycle \((1, 2, 3, ..., p)\). Define an action from \(H \times X\) to \(X\) by \(gx = g(x)\). This action just shuffles the elements of a \(p\)-tuple by shifting each element to the right \(1, 2, ..., \) or \(p\) times. Note that the order of \(H\) is \(p\).

<table>
<thead>
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<th>Example Continued:</th>
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</thead>
<tbody>
<tr>
<td>Let (G = D_6) and (p = 3).</td>
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<tr>
<td>C. Find an element with an orbit containing 3 elements. List its orbit.</td>
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<tr>
<td>D. Find all elements with orbits containing only 1 element.</td>
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</tbody>
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Let $G$ be any finite group whose order is divisible by the prime $p$. With the action described above answer the following:

1. How many elements are in $X$? Explain.

2. Which elements are in orbits all by their lonesome? What does $|X_H|$ represent in this context? Explain.

3. Since $H$ has order $p$, Theorem 36.1 says that $|X| \equiv |X_H| \pmod{p}$. What does this tell us about the number of elements of order $p$ in $G$. Explain.
II. Lemma (36.6)

**Timeout for Notation**

For any group $G$, we can define an action on the set, $X$, of all subgroups of $G$ by conjugation.

In this case, the subgroup $G_H = \{ g \in G : gHg^{-1} = H \}$ is called the *normalizer* of $H$ in $G$. The normalizer of $H$ is the largest subgroup of $G$ in which $H$ is normal.

Any subgroup is a normal subgroup of itself. So the normalizer of $H$ always at least contains the elements of $H$. If the normalizer is all of $G$ then $H$ is a normal subgroup of $G$.

The normalizer will be denoted by its standard notation $N[H]$ rather than the group action specific notation $G_H$.

**Lemma 36.6** says that if $H$ is a $p$-subgroup of a finite group $G$ then

$$[G : H] \equiv [N[H] : H] \pmod{p}$$

Terminology: A $p$-subgroup is one of order $p^n$.

**Something That Will Come In Handy:**

Prove that if $G$ is finite then $g$ is in $N[H]$ if $ghg^{-1} \in H$ for every $h$ in $H$.

[What you are showing is that in the finite case $gHg^{-1} \subseteq H$ implies that $gHg^{-1} = H$]
Our goal: Prove Lemma 36.6, which says that of $H$ is a $p$-subgroup of a finite group $G$ then $[G:H] ≡ [N[H]:H](\text{mod } p)$

Let $G$ be a group, $H$ a $p$-subgroup, and $X$ the set of left cosets of $H$ in $G$ and let $H$ act on $X$ by left translation: $h(gH) = (hg)H$.

The plan is to use the super theorem 36.1 to prove $[G:H] ≡ [N[H]:H](\text{mod } p)$. We will be able to apply it (with $H$ as the group) since the order of $H$ is $p^n$.

Theorem 36.1 says that $|X| ≡ |X_G|(\text{mod } p)$. Observe that $|X| = [G : H]$ in this case.

If it turns out that $X_H$ is the set of cosets of $H$ in $N[H]$ then we are home free! Recall that $X_H$ is the set of elements of $X$ that are in orbits all by their lonesome.

Prove that $gH$ is in an orbit by itself iff $g$ is in $N[H]$

That does it! By Theorem 36.1, we know that for any finite group $G$ and any $p$-subgroup $H$ of $G$, $[G:H] ≡ [N[H]:H](\text{mod } p)$
III. First Sylow Theorem

Theorem 36.8 (First Sylow Theorem): Let $G$ be a finite group and let $|G| = p^n m$ where $n \geq 1$ and $p$ does not divide $m$. (This means that $p^n$ is the largest power of $p$ that divides the order of $G$.) Then $G$ contains a subgroup of order $p^i$ for each $i$ where $1 \leq i \leq n$.

[Note there is a second part in the book. We’ll come back and get it later if needed.]

This proof doesn’t use any cool group actions $\otimes$. It is induction proof using Cauchy’s theorem.

Proof:

Base Case: $i = 1$. This is just Cauchy’s Theorem. Nothing (left) to prove there!

Inductive Step: Suppose that $i < n$ and we have a subgroup, $H$, of order $p^i$.

Then $p$ divides $[G : H]$

Then by Lemma 36.6...

Now since $H$ is a normal subgroup of $N[H]$ we can form the group $N[H]/H$ and we see that $p$ divides the order of this group.

Here is how you do the rest of the proof [fill in the details in the box on the next page]:

Step 1: Apply Cauchy’s Theorem to $N[H]/H$.
Step 2: Examine the pre-image of this of the subgroup you get in Step 1 under the canonical homomorphism from $N[H]$ to $N[H]/H$ (this map just sends $a$ to $aH$).
Fact Needed For Step 2:

Prove that if $\alpha$ is a homomorphism from a group $G$ to a group $G'$ then the pre-image $\alpha^{-1}(H')$ of a subgroup $H'$ of $G'$ is a subgroup of $G$.

Cauchy's Theorem says $N[H]/H$ has a subgroup of order $p$ (Oops! I just did Step 1).

Call this subgroup $K$. What can we say about the pre-image of this subgroup? What is its order?

Hints:

- Given a group $G$ and a normal subgroup $H$, what exactly is the pre-image of an element under the canonical mapping?

- Still considering the canonical homomorphism from finite group $G$ to $G/H$, how many elements would be in the pre-image of a set of $m$ elements (still canonical mapping)?
An Example To Illustrate The First Sylow Theorem:

Remember when we were trying to figure out how many groups there are of order 12?

Well, since $12 = 2^2 \times 3$, Cauchy's Theorem says that we must have an element of order 2 and an element of order 3.

[You may recall that we proved we had to have an element of order 2 in any group of even order using the "inverse buddy pairs proof". Then we proved that we had an element of order 3 by using coset arguments.]

For one homework problem, we assumed that we had a normal subgroup of order 4. It turns out that there does have to be a subgroup of order 4, but it need not be normal. We can use the strategy of the proof of The First Sylow Theorem to show that a group of order 12 must have a subgroup of order 4.


Now by Lemma 36.6 we know that:

Note that if $H$ is normal in $G$, then $N[H]$ is $G$. Otherwise, $N[H]$ is a subgroup of order 6 or 4. It can't be order 2 since 2 divides $[N[H] : H]$. It is going to turn out that in this case it must be of order 4.

Again following the idea of the proof of The First Sylow Theorem, we observe that $H$ is normal in $N[H]$ and form the group $N[H] / H$. The order of this quotient group is divisible by 2, so we know it has a subgroup of order 2 by Cauchy's Theorem. This means that we have a subgroup consisting of the coset $<a>$ and a coset $b<a>$ where $b$ is in the normalizer of $H$.

The canonical map will take __________ to $<a>$ and it will take __________ to $b<a>$. Then ________________ is our subgroup of order 4!

For fun, we can use the fact that $\{<a>, b<a>\}$ is a subgroup to verify that this is a subgroup. We know that $<a> b<a> = b<a>$. Then since $ab$ is in $<a> b<a>$, we know it is in $b<a>$ (so it is either $b$ or $ba$). Since $b <a> b<a> = <a>$ and $b^2$ is in $b <a> b<a>$ we know that $b^2$ is in $<a>$. Thus $b^2 = e$ or $b^2 = a$. [This should all seem eerily familiar.] Anyhow, this is enough to verify that $\{e, a, b, ba\}$ is closed and hence is a subgroup.