1. Exam Monday November 14 (in class part).

2. Homework #5: (You will turn this in Nov 14 and we will grade two of the bold problems.)

Section 11: 46, 47, 49, 50, 51
Section 13: 49, 50, 51, 52.

3. Theorem: $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and (and so isomorphic to $\mathbb{Z}_{mn}$) if and only if $m$ and $n$ are relatively prime.

Proof: ($\Leftarrow$) Suppose $m$ and $n$ are relatively prime. Then the least common multiple of $m$ and $n$ is $mn$. Consider the element $(1, 1)$. Then $k(1, 1) = (k, k) = (0, 0)$ if and only if $k$ is a multiple of both $m$ and $n$. Since $mn$ is lcm$(m,n)$, it is the smallest “power” of $(1, 1)$ that equals the identity element. Thus $|(1, 1)| = mn$ so that $<(1, 1)>$ is a cyclic subgroup that is the same size as the full group. Therefore, it is the whole group and $\mathbb{Z}_m \times \mathbb{Z}_n$ and so $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic.

($\Rightarrow$) Now suppose that gcd$(m,n) = d > 1$. Then $mn/d$ is less than $mn$ and is a multiple of both $m$ and $n$ (it is the lcm). Now let $(r, s)$ be an element of $\mathbb{Z}_m \times \mathbb{Z}_n$ and note that $(mn/d)(r, s) = ((mn/d)r, (mn/d)s) = (0, 0).$ So all elements of $\mathbb{Z}_m \times \mathbb{Z}_n$ have order less than $mn$ (the order of the full group), so none can generate the group.

4. Corollary: The group $\prod_{i=1}^{n} \mathbb{Z}_{m_i}$ is cyclic if and only if gcd$(m_i, m_j) = 1$ for all $1 \leq i, j \leq n$. 
5. Fundamental Theorem of Finitely Generated Abelian Groups:
Every finitely generated abelian group $G$ is isomorphic to a direct product of cyclic groups of the form $\prod_{i=1}^{n} \mathbb{Z}_{(p_i)^{r_i}} \prod_{j=1}^{m} \mathbb{Z}$ where the $p_i$ are primes (not necessarily distinct) and the $r_i$ are positive integers. And this product is unique except for possible rearrangement of the factors.

Proof: Omitted from the book. You can find one in Dummit & Foote, but it requires Module Theory.

6. The FTFGAG can help to classify finite groups.

7. What are all the abelian groups of order 128? How about order 120?

$128 = 2 \times 2 \times 2 \times 2 \times 2 \times 2$. So we have $\mathbb{Z}_{128}$, $\mathbb{Z}_{64} \times \mathbb{Z}_{2}$, $\mathbb{Z}_{32} \times \mathbb{Z}_{4}$, $\mathbb{Z}_{32} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}$, $\mathbb{Z}_{16} \times \mathbb{Z}_{8}$, $\mathbb{Z}_{16} \times \mathbb{Z}_{4} \times \mathbb{Z}_{2}$, $\mathbb{Z}_{16} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}$, $\mathbb{Z}_{8} \times \mathbb{Z}_{8} \times \mathbb{Z}_{2}$, $\mathbb{Z}_{8} \times \mathbb{Z}_{4} \times \mathbb{Z}_{2}$, $\mathbb{Z}_{8} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}$, $\mathbb{Z}_{4} \times \mathbb{Z}_{4} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}$, $\mathbb{Z}_{4} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}$, and finally $\mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}$.

$120 = 2^3 \times 3 \times 5$. So we have $\mathbb{Z}_{8} \times \mathbb{Z}_{3} \times \mathbb{Z}_{5}$, $\mathbb{Z}_{4} \times \mathbb{Z}_{2} \times \mathbb{Z}_{3} \times \mathbb{Z}_{5}$, and $\mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{3} \times \mathbb{Z}_{5}$.

Section 13: Homomorphisms

1. Recall that a homomorphism, $\varphi$, is a map from a group $G$ to a group $H$ with the property that for all $a, b$ in $G$, $\varphi(ab) = \varphi(a)\varphi(b)$.

2. Suppose $G$ is abelian and $\varphi$ is a homomorphism from $G$ to $H$. What additional property of $\varphi$ is sufficient to ensure that $H$ is abelian?

We left off with the most popular conjecture being “onto”