CONJECTURING AND PROVING AS PART OF THE PROCESS OF DEFINING

Sean Larsen
Portland State University
slarsen@pdx.edu

Michelle Zandieh
Arizona State University
zandieh@asu.edu

Researchers and mathematicians have argued that students should be engaged in the activity of defining mathematical concepts. This report looks at the role of proving in students’ defining activity. A preliminary framework is offered to account for the ways in which proving can contribute to the process of defining. Three categories of contribution (motivation, guidance, and assessment) are illustrated in the context of two classroom episodes (one from a geometry course and another from a group theory course) in which students are engaged in defining.

Introduction

Freudenthal (1973) noted that definitions are generally not preconceived but are just the finishing touches of the mathematical activity of defining. He argued that students should not be denied the opportunity to participate in this activity. De Villiers (1998) also argued that students should be actively engaged in the process of defining in order to highlight the meaning of the content and to allow students to actively participate in the construction and development of the content. He further suggested it might be essential to engage students in the process of defining in order to increase their understanding of definitions and the concepts to which they relate.

De Villiers (1998) pointed out that defining is inherently a complex mathematical activity. Zandieh and Rasmussen (in preparation) take defining to include not just formulating a definition but also activities such as negotiating and revising a definition. These activities may involve generating conjectured definitions, creating examples to test the conjectures, and trying to prove whether or not a conjectured definition “works” in the sense of doing the job that the definition is being created to do. Zandieh and Rasmussen include these activities as part of what they mean by defining. They also note that the defining process includes negotiating both the way the definition should be formulated and the deeper issue of what the concept should mean.

While the research literature suggests the value of engaging students in the activity of defining, there is still much we need to know in order to support this kind of mathematical activity. De Villiers (1998), drawing on Freudenthal (1973), elaborated the activity of defining by describing the categories of constructive and descriptive defining. Descriptive defining involves singling out some properties of a well-known object while constructive defining involves creating new objects from familiar ones. Zandieh and Rasmussen (in preparation) added additional structure to the activity of defining by considering students’ defining in terms of Gravemeijer’s (1999) levels of mathematical activity (in the task setting, referential, general, and formal). Additionally, researchers have conducted teaching experiments in order to develop instructional sequences and instructional theories to support defining in particular content areas (e.g., De Villiers, 1998; Larsen, 2004, Rasmussen, Zandieh, King, & Teppo, 2005).

The goal of this paper is to further explore and elaborate the activity of defining by analyzing the role of proving as students develop definitions in the context of college geometry and group theory. Our focus on this aspect of defining activity is inspired by Lakatos’ (1976) case studies that illustrated the historical development of mathematical concepts through a process of “proofs and refutations” – a process that featured movement back and forth between proving and
defining. We have observed this same phenomenon in our instructional design research at the undergraduate level. By analyzing the role of proving in students’ defining activity in different content areas, we aim to develop a better theoretical understanding of the interaction of these important aspects of mathematical activity. In turn we expect this understanding to guide the development of instructional theory and practice in support of students’ mathematical activity.

**Research Methods**

The data for this research comes from teaching experiments (Cobb, 2000) in college geometry and elementary group theory. While these teaching experiments varied in many respects, they both included the development of instructional approaches designed to support mathematical learning through the students’ own mathematical activity. In particular, each of these teaching experiments involved engaging the students extensively in defining. Data consisted of videotapes of all class sessions and photocopies of the students’ written work. The data analysis consisted of multiple phases of iterative analysis (Cobb & Whitenack, 1996).

In the following sections we present two episodes in which students were engaged in developing definitions for specific mathematical objects. The first episode is situated in a college geometry course. The students were engaged in defining a special class of triangles on the sphere (small triangles) for which the Side-Angle-Side (SAS) congruence theorem is valid. The second episode is situated in a group theory course. The students were engaged in developing an improved definition of subgroup. Each of these defining activities is an example of constructive defining because they involve the creation of new objects from familiar objects. Furthermore in each case there was an original definition for the students to work with as they developed the new definition. The geometry students had already developed a definition of triangle that makes sense on the sphere while the group theory students had agreed that a subgroup is a subset of a group that is also a group under the same operation. However, there were significant differences between these two defining activities as well. In the geometry episode, the students were developing a new class of spherical triangles by adding more restrictive conditions to the existing definition. The main criterion for evaluating this new definition was that SAS holds for the new class of triangles. In the group theory episode, the students were developing an alternative but equivalent definition of subgroup. The main criterion for evaluating this new definition was that it be equivalent but more efficient than the original definition.

**Geometry Episode: Defining a Class of Triangles for which SAS is True**

**Background**

The class had previously developed a definition for triangle on the surface of the sphere and had begun to explore properties of these triangles. During the class session discussed below, the students were working on proving SAS on the plane and determining whether or not SAS was true on the sphere. If SAS was not true for all triangles on the sphere (which it is not), the students were to prove that by coming up with a counterexample. The students were finally asked to find a subset of triangles on the sphere, labeled by the textbook (Henderson, 1996) as “small triangles,” for which the theorem is true.

**Proof as Motivation and Guidance for Defining**

The students’ experience the previous day with a wide variety of unexpected triangles on the sphere had given them a sense of what might be meant by a small, medium or large triangle --the small triangles being the ones that look most like planar triangles. So, student defining of “small
“triangle” was influenced in part by this preordained label. However, most of the student discussions of potential small triangle definitions were focused on issues that are related to proving: proving (finding a counterexample) that SAS is false in general for triangles on the sphere and proving that SAS is true for all triangles on the plane.

Early in the small group discussion the students in the group readily found a counterexample for SAS by noting that the endpoints of two given sides of a triangle could be connected on the sphere by two different line segments thus creating two non-congruent triangles with a side, included angle, and side in common.

Amy: So we’ve got side-angle-side. And one side would be going around there and the other one would be all the way around back like that.

Sam: But they’re not congruent. So, it doesn’t work. You have to limit it and say that it only works for small triangles. That you can’t go all the way around the sphere.

Amy: And did we define small triangles?

Sam: No, but we probably need to.

Amy: Yeah. [pause] Small triangles is an area less than half the area of the sphere?

Note that Amy’s definition will eliminate one of the two triangles in the counterexample, but it also may simply have been an attempt to limit the size or area of the small triangles. At this point the students were distracted and did not return to the issue of defining small triangle until after working on the proof that SAS is true on the plane. The proof encouraged by their textbook involved using symmetries and transformations to line up the side, angle, and side that are given to be congruent. At this point the students had to use a “fact” discussed in class that, on the plane, two points (in this case the two end points of the given congruent sides) determine one and only one line. This means there is only one segment that can be the third side of the triangle. Therefore, the triangles are congruent because one has been made to lie on top of the other.

Amy: So this proof doesn’t work on a sphere?

Tom: No.

Sam: It must be because of the last part.

Amy: Right. So what do we do as far as proving it on the sphere?

Cindy: Because it doesn’t work.

Amy: But it does work for small triangles right?

Jay: You’ve got to define small.

Sam: As long as you defined a straight line to be the shortest distance, not just any distance from B to C. That’s why it falls apart in the sphere case because you can go outward from B and come inward on C. It could be the shortest distance. I mean maybe that’s the definition of a small triangle is if you have points A, B, and C they’re connected by a straight, shortest distance line.

Amy: Okay. Yeah. I like that too. I like that better. It’s more concrete.

Sam described a way to define small triangle that has a direct connection to both eliminating the counterexample and replacing the step in the proof that is true only on the plane. The defining was motivated by the counterexample and the proof on the plane, but these proofs also directly guided the creation of Sam’s definition. Note that Sam’s definition is more closely related to both the counterexample and the proof on the plane than Amy’s earlier definition.
Proof as a Way to Assess Defining

Immediately following the discussion above, the students began the process of assessing whether their definition would accomplish the purpose for which it had been created, i.e. assessing whether, with this definition, they could prove SAS was true for small triangles.

Amy: So how do we prove that it does work for small circles [sic] on the sphere if we can’t use reflection symmetry? [pause] Rotation?

Tom: Couldn’t we do it if we’re working with small triangles? Can’t we use reflection symmetry?

Cindy: We’re gonna have to prove, like he said yesterday—We’re going to have to take the image, rotate, reflect—

Jay: Doesn’t this proof work for small triangles anyway?

Amy did not answer Jay directly but continued to push the group to confirm that the transformations and symmetries involved in the planar proof would work on the sphere as well. Then the teacher moved the class back to whole class discussion before the small group reached the point in the proof for which the small triangle definition is needed. Even though Sam had mentioned in the previous episode that the “last part” of the planar proof was probably the part that failed on the sphere, Amy made a point to assess whether each part of the proof would work on the sphere using the new small triangle definition.

Episode Wrap-up

On the next day of class a number of possible definitions for small triangle were discussed with respect to whether they made the SAS congruence theorem true. As the course continued, these discussions extended to whether or not the various definitions were equivalent and whether or not each would allow for the Angle-Side-Angle theorem to be true.

Group Theory Episode: Developing a More Useful Definition of Subgroup

Background

The class had previously developed a definition for group and agreed (after some discussion) that a subgroup would be a subset of a group that was a group under the same operation. In the class session described in the following episode, a group of three students was engaged in the process of developing a more useful definition for subgroup. Proof as Motivation for Defining

As the defining process began, the students had a working definition of subgroup as a subset that is a group under the same operation. Although this definition is quite natural, it is somewhat inconvenient to use in practice because in order to prove a subset is a subgroup it is necessary to verify that the subset satisfies all aspects of the group definition (closure, associativity, identity, inverses). Thus the motivation for developing an improved definition was the desire for a definition that made it easier to prove that a subset is a subgroup. The goal for the students was to determine the smallest number of properties that would need to be verified in order to determine whether a subset was a subgroup. As it turns out, it is only necessary to check that a subset is 1) non-empty, 2) closed under multiplication, and 3) includes the inverse of each of its elements. So, it is not necessary to verify associativity or the existence of an identity element.

The process began with Steve making a conjecture that it was only necessary to check that a subset is closed under the operation in order to show that it is a subgroup.

Phil: Closure, and after closure…

Steve: I think it's just closure.

Mike: You only need to check closure as long as you know it’s a subset of a group.
**Proof as a Way to Assess Defining**

After a conjectured definition had been proposed, the students were able to assess it by attempting to prove that it was equivalent to the original definition. In the following excerpt, Phil outlines a proof that closure is sufficient to show a subset is a subgroup. Note that although Phil’s attempt to take care of the infinite case was unsuccessful, the remainder of his argument works in the finite case. So, while this proof attempt verified that it is sufficient to check closure in the finite case, it did not allow the students to successfully evaluate the conjecture that closure is sufficient in general, because the students did not notice the subtle flaw in the proof. (While it is true that even in the infinite case each element appears exactly once in each row and column of a group’s operation table, this property is not necessarily inherited by closed subsets.) However, the teacher was able to offer a counterexample to more fully assess this conjecture.

Phil: Closure means each element appears exactly once.
Mike: Closure says each element appears exactly once? Exactly once in what?
Phil: If something is closed and you have a finite set then basically every element of that set is in the row.
Mike: Not necessarily.
Teacher: So what if it’s not a finite group?
Phil: Well, we already proved it for the infinite case that each element will appear exactly once in each row and column. So if we know it’s going to appear exactly once in each row or column then we can make x the arbitrary element which means a times the arbitrary element still guarantees that a is somewhere in there. So if we solve for x then x would have to be I. And then if we know I is in the group then we can basically say a times some arbitrary element will still give me I in the group, and then if you solve for x…
Teacher: So what are you even trying to define or prove here?
Phil: I'm trying to prove you only need closure.
Teacher: So consider the following example: real numbers under addition. Is that a group?
Phil: Yeah.
Teacher: Now, consider the following subset: positive numbers under addition.

**Proof as a Guide for Defining**

The counterexample offered by the teacher was also useful as a guide for the ongoing development of the definition. In the discussion shown below, the students analyzed this counterexample with the goal of improving their definition. The result was a new conjectured definition: a subgroup is a closed subset of a group that contains the inverse of each of its elements.

Phil: I forgot to say it has to have the same group operation.
Teacher: I didn't change the operation.
Mike: It's not closed.
Teacher: Are you sure?
Phil: Not a subgroup because don't have inverses.
Teacher: You didn't say I had to have inverses. You said I only had to be closed.
Steve: He's right.
Phil: Trying to think of a way around it.
Steve: So it’s inverses and closure.
Episode Wrap-up

The discussion continued for a while longer. A number of conjectured definitions were offered up by the students and then rejected as they developed counterexamples. Eventually Phil offered a further refinement of Steve’s original conjecture that closure would be sufficient, and Mike offered an improved version of the desired alternative definition of subgroup.

Phil’s new conjecture: “If you're talking about an infinite group you were talking about finite groups before so maybe there’s a couple different cases. If it’s finite then you only need closure.”

Mike’s new conjecture: “But you can get identity from the inverse law I think. The inverse law says that for all $a$ in $S$ there exits an inverse in $S$ such that it makes $I$, and that also has to be the inverse… and from that you can derive that $ai = a$ from this.”

Following this small group discussion Mike’s conjecture was discussed during a whole class discussion. The condition that the subset be non-empty was added in order to make a proof (that the new definition was equivalent to the original definition) work. Finally a subgroup was defined as a nonempty subset of a group having the properties that 1) it is closed under the operation and 2) the inverse of every element of the subset is also in the subset. Later in the course the students were asked to prove Phil’s conjecture, that in the finite case it was sufficient to check closure, using this definition.

Conclusions

In the two episodes, the students’ proving activity contributed in a variety of ways to their defining activity. As an initial step toward developing a framework for making sense of the role of proving in students’ defining activity, we categorize some of these contributions as follows.

Proof as Motivation for Defining

Proving served a motivational role in both of the episodes. The purpose of creating a definition for small triangle is to make it possible to prove (on the sphere) a theorem that is true and extremely useful in the plane. The purpose of creating an improved definition of subgroup is to make it easier to prove that a given subset is a subgroup. From an educational perspective, the fact that proving can motivate defining should not be overlooked. This aspect of defining activity can allow students to see that “defining is more than describing, that it is a means of the deductive organization of the properties of an object.” (Freudenthal, 1973, p. 417).

Proof as a Guide for Defining

Proving also served as a guide for defining in both episodes. In the geometry episode, the proof of SAS on the plane coupled with the proof (counterexample) that SAS is not true in general on the sphere suggested that a successful definition would not allow each pair of vertices to be connected by a side in two ways. In the group theory episode, the proof (counterexample) that closure was insufficient in general suggested that the existence of inverses might be particularly important. This particular contribution to students’ defining activity has been particularly powerful in our developmental research. For example we have found that reflecting on the process of proving that two specific groups are isomorphic provides a great deal of guidance as students define isomorphism (Larsen, 2004; Weber & Larsen, in press).

Proof as Way to Assess Defining

Proving also provided a way for students to assess their defining as they went along and a way to evaluate their final definition. In the geometry episode, students were able to evaluate their definition by considering whether it made it possible to modify the planar proof of SAS to
work on the sphere. In the group theory episode, students were able to evaluate their definitions by proving whether they were equivalent to the old definition.

Informally, these three categories of contribution could be restated as follows. The role of proof in defining is to 1) tell you what job the definition needs to do, 2) suggest what the definition ought to look like in order to do that job, and 3) to let you determine whether it actually does the job it is supposed to do.

Final Comments

The purpose of the framework of categories described in the previous section is to 1) add structure to our understanding of the process of defining by highlighting the important ways that proving can contribute to students’ defining activity and 2) to inform instructional design and practice by identifying the ways that proving can be leveraged to support students’ defining activity and concept development.

Finally, one of the goals of mathematics education research is to improve students’ understanding of and ability to construct proofs. Much of students’ difficulty with proof can be tied to their understanding and use of definitions. The research literature makes it clear that students’ concept images are not well connected to definitions for concepts and that they struggle to use definitions in proofs (e.g., Edwards & Ward, 2004; Moore, 1994; Tall & Vinner, 1981). Evidence is emerging that suggests that engaging students in defining can help them overcome some of these difficulties (Weber & Larsen, in press).

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