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Reviewed work(s):
Source: Journal for Research in Mathematics Education, Vol. 43, No. 4 (July 2012), pp. 465-493
Published by: National Council of Teachers of Mathematics
Stable URL: http://www.jstor.org/stable/10.5951/jresematheduc.43.4.0465
Accessed: 02/10/2012 16:29

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Coming to Understand the Formal Definition of Limit: Insights Gained From Engaging Students in Reinvention

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The purpose of this article is to elaborate Cottrill et al.’s (1996) conceptual framework of limit, an explanatory model of how students might come to understand the limit concept. Drawing on a retrospective analysis of two teaching experiments, we propose two theoretical constructs to account for the students’ success in formulating and understanding a definition of limit. The first construct relates to the need for students to move away from their tendency to attend first to the input variable of the function. The second construct relates to the need for students to overcome the practical impossibility of completing an infinite process. Together, these two theoretical constructs build on Cottrill et al.’s work, resulting in a revised conceptual framework of limit.

Key words: Advanced mathematical thinking; Calculus; Cognitive development; College mathematics; Reasoning

Many scholars hold the position that the concept of limit is fundamental to the study of calculus and analysis (e.g., Bezuidenhout, 2001; Cornu, 1991; Dorier, 1995). In particular, this concept is fundamental to a formal study of calculus grounded in standard analysis. Cornu observes that limit “holds a central position which permeates the whole of mathematical analysis—as a foundation of the theory of approximation, of continuity, and of differential and integral calculus” (p. 153). Indeed, limits arise in these and many other mathematical contexts, including the convergence and divergence of infinite sequences and series, and mathematical descriptions of the behavior of real-valued functions. The formal definition of limit at a point

\[
\lim_{x \to a} f(x) = L \text{ if for every } \varepsilon > 0, \text{ there exists a } \delta > 0, \\
\text{such that } 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon
\]

There is an equivalence class of definitions of limit, all of which we consider to be “the” formal definition of limit. Use of a formal definition of limit might lead the reader to believe that we had a specific definition in mind that we wanted the students to reinvent. Thus, we use the formal definition of limit throughout the article.

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is foundational as students proceed to more formal, rigorous mathematics. Continuity, derivatives, integrals, and Taylor series approximations are just a few of the topics in an analysis course for which the formal definition of limit serves as an indispensable component. Further, this definition often serves as a starting point for developing facility with formal proof techniques, making sense of rigorous, formally quantified mathematical statements, and transitioning to abstract thinking (Ervynck, 1981; Tall, 1992). For all these reasons, the formal definition of limit holds an important place in pedagogical considerations and as an object of research in mathematics education.

Relatively little is known about how students might come to reason coherently about the formal definition of limit as they progress to more advanced courses. Although some authors have suggested pedagogical approaches for teaching students about the formal definition of limit (e.g., Gass, 1992; Steinmetz, 1977), and others have suggested that the formal definition of limit is too cognitively sophisticated for introductory calculus students (Cornu, 1991; Dorier, 1995; Tall, 1992; Vinner, 1991), the overarching objective of our research was to move toward the elaboration of a cognitive model of what coming to understand the formal definition of limit might entail. Guiding our work was the following research question:

In the process of generating a precise definition of limit, what challenges do students experience, and how are such challenges resolved?

Swinyard (2011) provides a descriptive account of the first of two teaching experiments, during which a pair of students reinvented a formal definition of limit. Two themes that emerged in this account were: (a) students were inclined to focus their attention first on the $x$-axis, a perspective incongruent with the process described by the formal definition; and (b) students struggled to characterize what it means for a function to be infinitely close to a point. In this article, we build on Swinyard’s descriptive account in order to elaborate and extend Cottrill et al.’s (1996) conceptual framework of the limit concept. First, recognizing that finding limit candidates and verifying limit candidates involve distinct mental processes, we elaborate the need for students to shift from the $x$-first process used to identify limit candidates to a $y$-first process necessary for verifying that a given candidate is indeed a limit. Second, we describe ways of thinking that can be used to operationalize the notion of infinite closeness. Prior to presenting our findings, and in an effort to provide appropriate context for the reader, we begin by situating this study within the literature on students’ understanding of the formal definition of limit.

STUDENTS’ UNDERSTANDING OF THE FORMAL DEFINITION OF LIMIT

Extensive research (e.g., Bezuidenhout, 2001; Davis & Vinner, 1986; Ferrini-Mundy & Graham, 1994; Monaghan, 1991; Tall & Vinner, 1981; Williams, 1991) delineates the common student misconceptions related to the limit concept. Most of this research has focused on informal conceptions of limit in the context of 1st-year calculus. A smaller body of research (Cornu, 1991; Cottrill et al., 1996;
Ervynck, 1981; Fernández, 2004; Tall, 1992; Tall & Vinner, 1981; Vinner, 1991; Williams, 1991) has addressed students’ understanding of the formal definition of limit. This research indicates that students generally are unable to communicate coherent understanding of the formal definition of limit for a variety of reasons. One source of difficulty that has been identified is that students struggle to make sense of the abundant algebraic notation present in the conventional $\varepsilon-\delta$ definition of limit (Cornu, 1991; Cottrill et al., 1996; Ervynck, 1981; Fernández, 2004). For example, Fernández reports that students expressed confusion about: (a) what $\varepsilon$ and $\delta$ represent; (b) the relationships between the variables (and parameters) in the definition; and (c) why $|x-c|$ is required to be positive, whereas $|f(x)-L|$ is not. Research also suggests that students’ difficulties with the formal definition of limit are partially attributable to the struggles they experience with quantification (Cottrill et al., 1996; Dubinsky, Elterman, & Gong, 1988; Tall & Vinner, 1981).

Although the research suggests that the formal definition of limit is difficult for students to understand, not much is known about how they might come to understand it. Cottrill et al. (1996) make a unique contribution in this regard. Rather than identifying types of difficulties experienced by students, the authors provide a cognitive model, called a genetic decomposition, which describes what it could mean to come to understand the definition of the limit concept. This genetic decomposition consists of seven steps, describing gradually more sophisticated mental constructions related to the limit concept. The first four steps are concerned with the informal limit concept and are focused on the cognitive processes involved in identifying the limit of a function at a point. The final three steps, which focus on the formal definition, are concerned with the formalization of these processes.

The goal of this article is to build on the field’s understanding of how students can come to understand the formal definition of limit. Although the framework proposed by Cottrill et al. (1996) provides a starting point, Cottrill et al. did not provide empirical support for the steps of their model focused on the formal definition. Additionally, we note that the processes involved in identifying limit candidates (the processes described in the first four steps of Cottrill et al.’s model) are different from the processes captured by the formal definition of limit (which involve verifying that a number is the limit). Thus, it may not be appropriate to conceptualize students’ learning of the formal definition in terms of the formalization of the processes involved in finding limits.

However, like Cottrill et al. (1996), we see the value in connecting students’ informal and formal understandings of limits. Fernández (2004) suggests that students may reason more coherently about the formal definition when allowed to build on their spontaneous conceptions. In particular, she notes that when students in her study were not forced to use notation traditionally associated with the definition, they were more likely to reason coherently about the formal limit concept. Similarly, Tall and Vinner (1981) observed that students’ concept images of limit (which typically are dominated by ideas related to finding limits) tend to be
disconnected from the concept definition of limit. They further note that students are unlikely to rely on the definition when reasoning about limits; instead, they are likely to rely on their concept images. Our goal is to build on Cottrill et al.’s genetic decomposition by describing the cognitive processes involved in students developing an understanding of the formal definition that is connected to their concept images. Swinyard (2011) provides an existence proof and a descriptive account of how students can reinvent the formal definition of limit. In the current article, our aim is to build on Swinyard’s descriptive account in order to elaborate and extend Cottrill et al.’s (1996) conceptual framework of the limit concept.

**AN EXISTING MODEL OF STUDENT REASONING**

Cottrill et al. (1996) provide a genetic decomposition of the limit concept that describes a sequence of mental constructions that students might make in the process of coming to understand the limit concept, both informally and formally (Figure 1).

Cottrill et al. (1996) were able to find empirical evidence for the first four steps of this genetic decomposition and were able to refine these aspects based on their analyses of the participating students’ reasoning. Because the students in the Cottrill et al. study showed almost no evidence of reasoning at the levels described by Steps 5–7, the latter steps of the genetic decomposition remain unsubstantiated empirically.

1. The action of evaluating \( f \) at a single point \( x \) that is considered to be close to, or even equal to, \( a \).
2. The action of evaluating the function \( f \) at a few points, each successive point closer to \( a \) than was the previous point.
3. Construction of a coordinated schema as follows.
   a. Interiorization of the action of Step 2 to construct a domain process in which \( x \) approaches \( a \).
   b. Construction of a range process in which \( y \) approaches \( L \).
   c. Coordination of (a), (b) via \( f \). That is, the function \( f \) is applied to the process of \( x \) approaching \( a \) to obtain the process of \( f(x) \) approaching \( L \).
4. Perform actions on the limit concept by talking about, for example, limits of combinations of functions. In this way, the schema of Step 3 is encapsulated to become an object.
5. Reconstruct the processes of Step 3(c) in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, in symbols, \( 0 < |x - a| < \delta \) and \( |f(x) - L| < \varepsilon \).
6. Apply a quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of limit.
7. A completed \( \varepsilon - \delta \) conception applied to a specific situation.

*Figure 1. Refined genetic decomposition of limit (Cottrill et al., 1996, pp. 177–178).*
Cottrill et al. hypothesized, however, that reasoning coherently about the formal definition of limit entails formalizing one’s informal understandings of limit. In the decomposition shown in Figure 1, doing so amounts to formalizing the first three steps, specifically by reconstructing the coordinated dynamic processes (as \( x \) gets closer to \( a \), \( f(x) \) gets closer to \( L \)) described in Step 3c into a logically formulated structure (if \( 0 < |x - a| < \delta \), then \( |f(x) - L| < \varepsilon \)). However, although we agree that the formal definition of limit can be seen as a formalization of a process, we argue that the process underlying the formal definition of limit is a different process from the one described by the first three steps of Cottrill et al.’s genetic decomposition. Instead, we suggest that the coordinated processes described in Step 3c of Cottrill et al.’s genetic decomposition characterize how one might find a candidate for the limit. Research (e.g., Cottrill et al., 1996; Swinyard, 2011) provides evidence that when students select a candidate for the limit of a function, they do so using an \( x \)-first perspective. By \( x \)-first perspective, we mean that students focus their attention first on the inputs (\( x \)-values) and then on the corresponding outputs (\( y \)-values). The selection of a candidate is made based on the numeric value (\( L \)) to which the \( y \)-values are getting close as \( x \)-values get closer to \( a \).

In contrast, the process captured by the formal definition is the process by which one would verify that a candidate is the limit of a function at a point, a process that requires one’s attention to be focused initially on the \( y \)-axis. Indeed, the formal definition of limit provides necessary and sufficient conditions that one can use to determine whether a proposed real number \( L \) is, in fact, the limit of a real-valued function \( f \) at a particular \( x \)-value. Research suggests that students miss this important distinction between finding and validating limit candidates (Bezuidenhout, 2001; Fernández, 2004; Juter, 2006). Validating a candidate via the formal definition of limit relies on one’s ability to reverse his or her thinking and use a \( y \)-first process. When we refer to a \( y \)-first process, we mean the process of validating a limit candidate by considering first a range of output values around the candidate and then subsequently determining whether there is an interval around the \( x \)-value of interest that will produce outputs (except possibly at \( x = a \)) within the preselected \( y \)-interval. When an individual approaches the limit concept by considering such a \( y \)-first process, we say he or she is reasoning from a \( y \)-first perspective. The conventional definition of limit is a formalization of this \( y \)-first perspective.

The genetic decomposition offered by Cottrill et al. (1996) has served as a helpful framework from which to develop our research. Specifically, based on our preceding arguments, we can identify a key question that needs to be addressed: How do students move from the \( x \)-first process described in Step 3 (as \( x \) gets closer to \( a \), \( f(x) \) gets closer to \( L \)) to the more formally expressed \( y \)-first process (for every \( \varepsilon > 0 \), there exists a \( \delta > 0 \), such that if \( 0 < |x - a| < \delta \), then \( |f(x) - L| < \varepsilon \)) described by the formal definition? The central objective of our research was to identify the challenges students experience in the process of generating a precise definition of limit, and how such challenges are resolved. Our efforts to address this objective have resulted in two specific findings presented subsequently in the Results section.
METHOD

The genetic decomposition proposed by Cottrill et al. (1996) is clearly intended to model the process of learning the limit concept in a way that features a strong link between students’ informal understanding and their understanding of the formal definition. Because we also are particularly interested in learning how students can understand the formal definition in a way that is well connected to their informal knowledge, we adopted a developmental research design. Gravemeijer (1998) describes the goal of developmental research as follows: “to design instructional activities that (a) link up with the informal situated knowledge of the students, and (b) enable them to develop more sophisticated, abstract, formal knowledge, while (c) complying with the basic principle of intellectual autonomy” (p. 279). More specifically, to achieve our research goal of understanding how students can develop a formal understanding of the limit concept that is well connected to their informal concept images, we drew upon the heuristic of guided reinvention. Gravemeijer and Doorman (1999) argue that “guided reinvention offers a way out of the generally perceived dilemma of how to bridge the gap between informal knowledge and formal mathematics” (p. 112). This dilemma is avoided because students are not presented with formal mathematics and expected to make connections with their informal knowledge. Instead, the students’ informal knowledge is taken as a starting point from which to develop the formal mathematics. These theoretical considerations support our decision to investigate students’ understanding of the formal definition of limit via engaging them in reinvention rather than by engaging them in recalling or interpreting the definition.

Gravemeijer, Cobb, Bowers, and Whitenack (2000) describe reinvention as “a process by which students formalize their informal understandings and intuitions” (p. 237). Traditionally, one approach to identifying a plausible instructional starting point has been to analyze the historical evolution of the mathematical topic with an eye toward identifying motivating problems. Our task design was guided by mathematics history in two ways. First, each teaching experiment was characterized by the pair of students actively building their knowledge of limit by conjecturing and refuting formulations of the definition in an iterative manner. Conjectured definitions were considered and refined through analysis of examples and nonexamples that students generated via their concept images. In this sense, the students’ interactions were reminiscent of the historical development of mathematical ideas through a process of proofs and refutations (Lakatos, 1976). For example, to facilitate this process, one of the early tasks involved having the students generate a collection of examples based on their concept images of limit in response to the following prompt:

Please draw as many distinct examples as possible of how a function could have a limit of 2 at \( x = 5 \). In other words, what are the different scenarios in which a function could have a limit of 2 at \( x = 5 \)?

Second, inspired by the historical development of analysis, we engaged the students in thinking about how they would verify that a number is a limit as a way to support their development of a definition of limit. Grabiner (1981) notes that historically the
development of suitable definitions required prior development of methods of proof. For example, inequalities did not appear first in Cauchy’s definition of limit but instead appeared first in his proofs of convergence. Based on this observation, a central strategy was to have students describe how they would verify that a number was a limit. For example, they were asked the following questions:

How would you justify that the limit is what you say it is? Under what conditions would you say that the graph of a function has a limit of 2 at \( x = 5 \)? What would have to be true about that function? Or what would have to be true about that graph?

The two teaching experiments were similar in that the central task was for the students to generate a precise definition of \( \text{limit at a point} \) that captured the intended meaning of the conventional \( \varepsilon-\delta \) definition. Instructional activities were focused primarily on discussing limits in a graphical setting, in hope that the absence of analytic expressions might support the enrichment of the visual aspects of the students’ respective concept images. The prototypical examples of limit generated by the students subsequently served as sources of motivation as they refined their definition. This approach was designed, in part, to enable us to build on the work of Cottrill et al. (1996) by exploring what might be involved in coming to understand the formal definition and how these understandings might build on the students’ concept images of limit. We argued previously that the standard formal definition of limit is a formalization of the process involved in validating that a given number is, in fact, a limit. Thus, one goal of our research was to understand how students might develop understandings of processes related to validating limit candidates and how they might then develop a formal definition based on these processes. It is in this way that the Cottrill et al. framework served as both a framework for analyzing our students’ mathematical activity (especially related to their informal understandings of limits) and for motivating and guiding our work with the students (in terms of task development).

**Data Collection Methods**

The study comprised two phases—the survey phase and the teaching experiment phase. Twelve volunteers from the first author’s Calculus III course completed a task-based survey assessing their informal reasoning about limit. Four of the students were selected for two teaching experiments, which formed the second phase of the study. Students were paired, and each pair participated in one teaching experiment. Each teaching experiment consisted of ten 60- to 100-minute paired sessions, and one 30- to 60-minute individual exit interview. Teaching experiments were conducted with pairs of students rather than individuals, with the intention of generating data regarding students’ reasoning in the form of naturally emerging discussions between the participating students in addition to data gathered from

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2 The first teaching experiment began just prior to the final exam for the Calculus III course. The second teaching experiment began 3 months after the conclusion of the Calculus III course.
Coming to Understand the Formal Definition of Limit

interactions between the researcher and the students. Instructional tasks used with the first pair of students were refined for use in the second teaching experiment. The paired sessions were conducted in a small classroom with the students responding to instructional tasks on the chalkboard in the front of the room. The first author conducted the teaching experiments serving as teacher–researcher, presenting the students with preconstructed tasks and asking follow-up questions to clarify their reasoning in response to tasks. Additionally, the teacher–researcher periodically asked the students questions designed to support their progress. For example, as a way to encourage the students to continue refining their definition during the first teaching experiment, he asked whether an early version of their definition of limit appropriately accounted for the functional behavior of a jump discontinuity.

The participating students, the researcher, and a research assistant were present for each session. Sessions were generally separated by a span of 6–10 days, allowing time for analysis between sessions and subsequent construction of appropriate instructional activities. All sessions, including the individual exit interviews, were video recorded by the research assistant. These 22 video recorded sessions were the primary source of data for the study. A chronological outline of the research study is given in Table 1.

Teaching Experiment Participants

The selection of the 4 students for the teaching experiment phase of the study was based on the extent to which they had developed and demonstrated strong informal understanding of limit throughout a 3-term introductory calculus sequence taught by the teacher–researcher. Specifically, this meant that students were able to: (a) identify and generate examples and nonexamples of limits (both at a point and at infinity), and (b) provide an informal definition of limit that is consistent with the idea of the

Table 1
Chronological Outline of the Research Study

<table>
<thead>
<tr>
<th>Phase</th>
<th>Research activity</th>
<th>Date</th>
</tr>
</thead>
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<td>Survey phase</td>
<td>Informal Limit Reasoning Survey</td>
<td>April 2007</td>
</tr>
<tr>
<td></td>
<td>Analysis of surveys and participant selection for teaching experiments (TE)</td>
<td>May 2007</td>
</tr>
<tr>
<td>Teaching experiment phase</td>
<td>TE #1 – Amy &amp; Mike</td>
<td>May–July 2007</td>
</tr>
<tr>
<td></td>
<td>Post analysis of TE #1 and refinement of instructional sequence</td>
<td>Aug.–Sept. 2007</td>
</tr>
<tr>
<td></td>
<td>TE #2 – Chris &amp; Jason</td>
<td>Sept.–Dec. 2007</td>
</tr>
<tr>
<td></td>
<td>Post analysis of TE #2</td>
<td>Jan.–May 2008</td>
</tr>
<tr>
<td></td>
<td>Retrospective analysis of data corpus</td>
<td>Ongoing</td>
</tr>
</tbody>
</table>

3 All student names are pseudonyms.
function values approaching the limit $L$ as the input values approach the point at which one is taking the limit (i.e., consistent with Step 3 of the genetic decomposition proposed by Cottrill et al., 1996). The selection of students with robust concept images of limit was purposeful—we aimed to select students for whom we could realistically expect to gather substantive insights into what challenges arise as students with strong concept images attempt to define limit. Evidence of the students’ informal understandings was drawn from both their participation in the calculus sequence and their responses to items on the Informal Limit Reasoning Survey used in the first phase of the study.

In addition to possessing a robust informal understanding of limit, a second criterion for selection was that participants had no previous experience with the conventional formal definition of limit. During the 3-term introductory sequence, the conventional formal definition was not presented in class and only appeared in an appendix of the course textbook (Stewart, 2001). In addition to learning standard algebraic techniques for determining limits, the students were informally introduced to the concept as the height (i.e., $y$-value) that a function “intends to reach” as the $x$-values approach a particular value $x = a$. The students also regularly were asked to analyze and generate graphical examples and nonexamples of limits. To further ensure that the participating students did not have prior experience with the formal definition, the Informal Limit Reasoning Survey included a question in which students were asked to describe what it meant for a function to have a limit at a point. None of the selected participants responded in ways that suggested that they had been introduced to the formal definition.

The teaching experiment participants included 1 female and 3 males, with ages ranging from 19 to 28. Additional background information is provided in Table 2.4

**Description of Data Analysis**

In this section, we describe three phases of analysis. First we describe the ongoing analysis that occurred both between teaching experiment sessions and in the midst

<table>
<thead>
<tr>
<th>Name</th>
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<th>Calculus I grade</th>
<th>Calculus II grade</th>
<th>Calculus III grade</th>
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<td>A</td>
<td>A</td>
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<tr>
<td>Mike</td>
<td>Mathematics</td>
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<td>Computer Science</td>
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<td>A-</td>
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<td>Jason</td>
<td>Philosophy</td>
<td>A</td>
<td>A-</td>
<td>Pass</td>
</tr>
</tbody>
</table>

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4 Jason chose to take the 3rd term of calculus under the pass/no-pass grade option because Calculus III was not a requirement for his major. His level of demonstrated understanding of the material in the third course was similar to that of his understanding demonstrated in the first two Calculus courses.
of teaching experiment sessions. Next, we briefly describe the post analysis that was conducted after each of the teaching experiments. Finally, we describe the retrospective analysis that is the focus of this article.

The ongoing analysis was consistent with what Cobb (2000) refers to as micro-cycles of development and research. Between sessions the teacher–researcher transcribed the video of each session, paying particular attention to (a) articulated thoughts that seemed to provide leverage for the students, (b) the voicing of concerns or perceived hurdles that needed to be overcome, and (c) signs of/causes for progress and revision of conjectured definitions. Project team meetings were held between sessions to analyze the transcriptions and then plan for the following session. Drawing on this analysis, the teacher–researcher composed a document prior to each session, outlining the objectives for the upcoming session and describing the tasks that would be employed along with the rationale for each task.

The following is an illustrative example (from the first teaching experiment) of how the ongoing analysis of student reasoning influenced task design. Analysis of data revealed that by the end of the sixth session, Amy and Mike had reached an impasse in their attempt to define limit precisely. Their characterizations of limit were cast using an x-first perspective for which they showed no indication of dissatisfaction. Further, their characterizations included a notion, infinitely close (e.g., “as x gets infinitely close to a”), that they recognized as problematic. In response to this impasse, we decided to launch Session 7 by asking the students to set aside consideration of limits at a point and tasked them with characterizing—and then defining—the concept of limit at infinity. For two reasons we thought characterizing limit at infinity might provide necessary support for defining limit at a point. First, because describing limit at infinity primarily requires characterizing closeness on the y-axis but not on the x-axis, we conjectured that this context would support shifting the students’ attention to the y-axis. Second, we felt that limit at infinity would be a less cognitively taxing context in which to find a suitable replacement for the problematic notion of infinite closeness.

Ongoing analysis also occurred in the midst of each session of the teaching experiments, with the teacher–researcher selecting instructional tasks and constructing follow-up questions based on an on-the-fly analysis of student reasoning. For instance, as the students worked to characterize limit at infinity precisely during the seventh session, Amy noted that their definition needed to exclude extraneous y-values in close proximity to L that were not the limit. To capitalize on Amy’s observation, the teacher–researcher attempted to give the students a tool that would support them in addressing the problem they had identified. Specifically, because Amy spontaneously had introduced absolute value notation earlier in the session, the teacher–researcher conjectured that defining what it would mean for the function values to be merely close to a proposed limit L might lead Amy and Mike to think about progressively restrictive definitions of closeness, and that they might subsequently shrink y-bands around the limit L and

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5 For more details about this shift in the instructional sequence, see Swinyard (2011).
use absolute value statements to notate those increasingly restrictive definitions. In this way, he thought they might be able to adequately operationalize the troublesome notion of *infinite closeness*, and thus move to a more formal construction of the definition. We discuss this instructional intervention in the next section.

Following the completion of each teaching experiment, the research team conducted a *post* analysis. Transcripts for each teaching experiment were reviewed in chronological order with attention paid to excerpts that shed light on students’ reasoning and/or illustrated significant challenges or marked progress. The primary outcome of each post analysis was a learning-trajectory document for each pair of students that traced the development of the students’ reasoning as well as the evolution of their definition of limit.

Finally, we conducted a *retrospective* analysis of the entire data corpus. We began by reviewing the learning-trajectory documents with the aim of situating the events of each teaching experiment “in a broader theoretical context, thereby framing them as paradigmatic cases of more encompassing phenomena” (Cobb, 2000, p. 326). The retrospective analysis was a cyclic process in which hypotheses about students’ reasoning were generated, reflected upon, and subsequently refined until increasingly stable and viable hypotheses emerged. For instance, the first pass of the retrospective analysis confirmed that students’ reliance on an $x$-first perspective and reluctance to employ a $y$-first perspective was, in fact, an issue. Further, we recognized that the students struggled to find a suitable alternative to the troublesome notion of infinite closeness. Additional passes through the data were made with the intention of understanding what supported the students in resolving these challenges. This was followed by a phase of analysis in which the second author and a third researcher challenged the teacher–researcher’s interpretations and proposed alternative explanations of students’ reasoning as a means of scrutinizing the teacher–researcher’s analyses to establish internal validity (a similar process occurred during both the ongoing and post analyses).

In the next section, we provide the results of our retrospective analysis. In particular, we describe two central challenges the students faced and discuss how those challenges were resolved.

**RESULTS**

The purpose of this article is to elaborate the latter steps of the genetic decomposition proposed by Cottrill et al. (1996) to provide a more complete framework for what it might mean to come to understand the limit concept. Our elaboration is driven by our analyses of the two teaching experiments we conducted. In these experiments, two central challenges arose: (a) students relied on an $x$-first perspective and were reluctant to employ a $y$-first perspective, and (b) students struggled to operationalize what it means to be infinitely close to a point. In the first two subsections, we illustrate these two challenges with excerpts from our data and then, in the final subsection, describe the resolution (or partial resolution) of these challenges during each teaching experiment. We have elected to focus primarily on
the first teaching experiment because Amy and Mike were able to make more progress in their efforts to operationalize the idea of being infinitely close to a point, allowing us to more completely articulate our findings related to that issue. We do, however, include substantial discussion of the second teaching experiment for two reasons: (a) to provide evidence that the central challenges were not unique to the first pair, and (b) to provide additional insight into how students might mathematize what it means to be infinitely close to a point. (For additional detail regarding the second teaching experiment, see Swinyard, 2009).

Amy and Mike’s resolution of the two aforementioned challenges supported them in their efforts to construct a precise definition of limit. Indeed, our analysis of the teaching experiment suggests that their success in constructing a precise definition appears to have been supported by (a) their ability to shift their reasoning from an x-first perspective to a y-first perspective, and (b) their use of what we refer to as an arbitrary closeness perspective to operationalize what it means to be infinitely close to a point. Our retrospective analysis of the students’ activity as they developed both a y-first and arbitrary closeness perspective helps us to address the question that grew out of our analysis of the genetic decomposition offered by Cottrill et al. (1996): How do students move from the x-first process described in Step 3 (as \(x \) gets closer to \(a\), \(f(x)\) gets closer to \(L\)) to the more formally expressed y-first process (for every \(\varepsilon > 0\), there exists a \(\delta > 0\), such that if \(0 < |x - a| < \delta\), then \(|f(x) - L| < \varepsilon\)) captured by the formal definition?

Amy and Mike’s reinvention of the definition of limit unfolded in roughly four phases. Table 3 provides an overview of the different reinvention phases, as well as the key tasks with which the students were engaged. For an in-depth description of the guided reinvention process and the evolution of Amy and Mike’s definition of limit, see Swinyard (2011).

**Reasoning From an x-first Perspective**

In our review of the literature, we described the distinction between finding limit candidates and validating limit candidates, suggesting that the x-first process described in Step 3c of Cottrill et al.’s (1996) genetic decomposition characterizes how one might find a candidate for the limit, whereas the y-first process captured by the formal definition characterizes how one could verify that a candidate is the limit of a function at a point. With this distinction in mind, during the teaching experiments we focused the students on the task of validating limit candidates. During the third session, the teacher–researcher drew a function with a removable discontinuity (Figure 2) and asked Mike and Amy how they might convince someone that the limit of the function at \(x = 5\) was 7. This particular graph was drawn because it illustrates the need for the limit concept, because the limit at \(x = 5\) is 7, yet 7 is not the value of the function at that point. The intent of the task was for Amy and Mike to describe the process they would use to validate a limit candidate. Their responses, however, were reminiscent of the process described in Step 3c of Cottrill et al.’s genetic decomposition (i.e., as \(x\) gets closer to \(a\), \(f(x)\) gets closer to \(L\)). Indeed, Amy and Mike both showed a strong initial preference for
Table 3
*Overview of Teaching Experiment 1*

<table>
<thead>
<tr>
<th>Phase</th>
<th>Sessions</th>
<th>Key tasks/Prompts</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Generation of graphical</td>
<td>1–4</td>
<td>1. Generation of distinct examples of how a function could have a limit of 2 at x = 5.</td>
</tr>
<tr>
<td>examples and initial</td>
<td></td>
<td>2. “In general, under what conditions would you say that a function has a limit of 2 at x = 5?”</td>
</tr>
<tr>
<td>(x-first) characterizations of limit at a point</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. Recognition that the notion of infinitely close is problematic</td>
<td>5–6</td>
<td>1. “Describe a procedure one could use to see if the y-values for a function were getting infinitely close to a specific y-value L.”</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2. “What would happen procedurally if the y-values were not getting infinitely close to L?”</td>
</tr>
<tr>
<td>3. Construction of definition of limit at infinity</td>
<td>7–8</td>
<td>1. Generation of distinct examples of how a function f could have (or fail to have) a limit of 4 as $x \to \infty$.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2. “Give a careful definition of what it means for a function f to have a limit of 4 as $x \to \infty$.”</td>
</tr>
<tr>
<td>4. Refinement of definition of limit at a point</td>
<td>9–10</td>
<td>1. “Comment on the difference in specificity between your final definition of limit at infinity and your current definition of limit at a point.”</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2. Finalization of definition of limit at a point.</td>
</tr>
</tbody>
</table>

*Figure 2. Removable discontinuity graph.*
reasoning from an $x$-first perspective. For example, Amy responded by saying, “I’ll show you that, that for any $x$-value you can give me, I’ll give you a $y$-value that, that as your $x$-values get closer to 5, my $y$-values get closer to 7.” Asked to describe how he would convince someone that the limit of a function (at 0) was 2, Mike said, “I would do as Amy did earlier and tell them, give me any $x$-value close to 0, as close as you can get to 0. I will plug it in and I will give you a $y$-value that’s just about 2.”

Amy and Mike’s $x$-first perspective persisted well into the teaching experiment. The following formulation, constructed during Session 6, illustrates $x$-first reasoning reminiscent of Step 3c of the genetic decomposition (Cottrill et al., 1996):

**Pair 1, Definition 4:** The limit $L$ of a function at $x = a$ exists if every time we look at the function more closely as we get infinitely close to $x = a$, it bears out the same pattern of behavior, i.e., looks to be approaching some $y$-value $L$ w/ no vertical gaps in the graph. (Session 6)

The two students in the second teaching experiment (Chris and Jason) also expressed their initial characterizations of limit from an $x$-first perspective.

**Pair 2, Definition 1:** $y$ takes on values closer to the limit in question as you take $x$-values closer to the point at which you’re evaluating the limit. (Session 2)

It is important to note that the mention of $y$-values first in Chris and Jason’s first definition is not an indication that they were reasoning from a $y$-first perspective. This definition is describing a process that begins by taking $x$-values closer to the point at which the limit is being taken, with the result that $y$ takes on values closer to the limit in question. As Chris and Jason made efforts to refine their characterization during Session 4, the robustness of this $x$-first perspective was evident, as illustrated by the following excerpt.

**Jason:** I’m picturing the definition box in a calculus book. It would look like this. You couldn’t just say a limit is, fill in the blank. It would say it exists when all these things are [trails off]. So here, we’ll do it. Number 1, $x$, let’s see here, as $x$

**Chris:** As $x$ approaches $a$, as $x$

**Jason:** It’s hard to get away from as $x$ approaches $a$, then $y$ is approaching $L$.

We were not surprised to see the students initially implement an $x$-first perspective. The vast majority of students’ mathematical experiences with functions prior to the study of calculus are from an $x$-first orientation. As we have mentioned previously, this $x$-first perspective is further reinforced when students’ first encounters with the limit concept in an introductory calculus course are focused on finding limit candidates. Given such a background, students’ use of an $x$-first perspective in initial characterizations of limit is understandable.
Recognizing That the Notion of “Infinitely Close” Is Problematic

The reader will recall that the phrase “infinitely close” was present in Amy and Mike’s fourth definition. As we will demonstrate, Amy and Mike attempted to operationalize this idea in a number of ways using infinite processes. As they did so, they recognized that this approach generated another problematic issue, the fact that one cannot complete an infinite process. For example, the students quickly recognized this when they considered using a zooming metaphor to capture the idea of infinite closeness.

I: What would have to happen in a hypothetical world—you can do anything you want—what would have to happen for that limit to actually be 8? Under what conditions would that limit be 8?

Mike: I would have to be able to zoom in infinitely. I can’t really comprehend what that would be, but,

I: Oh, like keep this process going forever, or whatever?

Mike: Yeah.

I: Like this zooming-in process?

Mike: And the function would have to approach that height, from both sides, a specific one height from both sides, as I did this infinitely. That’s the only way I could be sure . . . .

I: Okay. So you’re saying that if I could do that forever.

Mike: Umm-hmm.

I: . . . Okay. Anything to add to that, Amy?

Amy: I like it. Yeah. Do it forever and then I’ll be happy with the graph. (Session 4)

The infinite nature of the limiting process, and the seeming impossibility of carrying out such a process, was a central concern for both students as the teaching experiment progressed. Amy addressed this issue directly during the sixth session.

Amy: I don’t know, it seems like we keep dancing around some kind of concept that we have to talk about in a series of . . . analogies or hypothetical situations, you know? . . . This like . . . hypothetical situation in which you are doing something forever . . . . I guess like the first thing that leaps to mind for me is that we’re trying to parse out what we mean by, by these impossible processes that we’re describing for . . . knowing whether we have a limit.

I: And you’re saying impossible there, why?

Amy: Because you can’t zoom in forever . . . You can’t do something an infinite number of times . . . . Through the methods that we’ve been talking about, all we can do is . . . grind through endless iterations until we get tired of it and like I give up. And you know, like, all you can do is find . . . the level of examination which disproves your idea but you can’t ever get to where you can conclusively prove it through the methods we’ve been discussing. (Session 6)

Amy was cognizant that establishing the existence of a limit via an iterative zooming process would not ultimately allow her and Mike to adequately describe
the physical phenomena in a satisfactory manner. Indeed, her comments suggest that she recognized that she and Mike were faced with the problem of trying to describe an impalpable construct (infinite closeness).

Amy: I have a hard time getting too worked up over the language about what it means to zoom and what we’re looking for when we zoom when we have lurking in the back this presupposition that whatever that means to zoom, . . . we have to repeat that process an infinite number of times. (Session 6)

Likewise, the second pair of students also noted the challenge of characterizing the notion of infinite closeness.

Jason: And what we’re trying to do with all this confusing language is describe in words what that function does in the vicinity of $a$.

Chris: As $x$ gets closer to $a$ . . . . I don’t want to say close, because how close is close?

Jason: Fantastic point.

The preceding excerpt captures Chris and Jason’s recognition of the challenge involved with describing a function’s local behavior. Chris’s question, “How close is close?” became a focal point for them during the experiment. During the fourth session, the teacher–researcher asked them whether they interpreted the phrases being close and being close enough as synonymous or different.

Jason: They’re not the same because being close, you could always get closer and

Chris: Being close enough implies that you’re there.

Jason: Yeah. Yeah, there’s an infinite amount of closers between close and close enough.

Jason’s claim that there are “an infinite amount of ‘closers’ between close and close enough,” along with his belief that “you could always get closer” anticipated subsequent struggles Chris and Jason had in conceiving of what it would mean for a function $f$ to be infinitely close to a particular $y$-value, $L$. As their conversation continued, Jason became more emphatic about his suspicion that completing an infinite process is impossible.

Jason: I’m raising an objection now to the idea of close enough. I don’t think that there is close enough, because of the idea that there’s always a closer. So if there’s an infinite number of “closers” between close and close enough, how can close enough even exist? . . . The idea of close enough means we’re getting ever so much closer to it, and I don’t know if I think there’s ever a close enough to say okay, now you’re there. You’re at the limit. You’re close enough now. I’m not sure about that.

Jason’s comments suggest he was experiencing conflict about defining a construct (limit) that he believed needed to include the notion of being “close enough,” an idea he was struggling to operationalize.
Resolving Cognitive Challenges

In both teaching experiments, the students reached an impasse in their attempts to define limit precisely. In an attempt to reconcile inconsistencies between their characterizations and their concept images, both pairs of students iteratively refined their definitions, paying particular attention to articulating precisely a notion (infinite closeness) that they could not adequately mathematize. At this point, they showed no indication of dissatisfaction with an $x$-first perspective, and although they recognized the limitations of the idea of infinite closeness, noting the impossibility of completing an infinite process in a finite amount of time, they did not recognize a suitable alternative. In both teaching experiments, the teacher–researcher responded by implementing an instructional intervention designed to support the students in resolving these difficulties. Below, we describe how the instructional intervention employed in the first teaching experiment supported Amy and Mike in shifting to a $y$-first perspective and in operationalizing the notion of infinite closeness. We then briefly discuss the intervention used in the second teaching experiment, noting how the cognitive shifts made by Chris and Jason compare to those made by Amy and Mike.6

First teaching experiment. We conjectured that limit at infinity would provide a suitable context for mathematizing the notion of infinite closeness, because the students could focus their attention on a single axis (the $y$-axis), instead of attending to the notion of infinite closeness on both the $x$- and $y$-axes (as the definition of limit at a point requires). Additionally, we felt that the task of defining limit at infinity would support our efforts to shift the students’ attention to the $y$-axis. Instructionally, the task of defining limit at infinity was similar to defining limit at a point. Session 7 began with Amy and Mike generating a variety of examples of functions that had limits at infinity. They produced a constant function, a function with a horizontal asymptote, and an oscillating function with amplitude approaching zero (Figure 3).

The students were then prompted to “give a careful definition of what it means for a function $f$ to have a limit of 4 as $x \to \infty$.” After the students struggled for some time with the idea of infinite closeness, and produced a number of invalid and imprecise characterizations (see Swinyard, 2011), the teacher–researcher encouraged them to formulate a concise definition. Amy responded by saying that “there needs to be some interval from $a$ to $\infty$ where the function is continuous7 and, um, where the maximum distance between the $y$-values and $L$ show a pattern of decreasing as $x$ increases.” As she began to write this down, she stopped herself and made a significant observation.

6 For a more descriptive account of the intervention used in the second teaching experiment, see Swinyard (2009).
7 The requirement of continuity mentioned here and in the following excerpt was not pursued by the students and did not ultimately appear in the students’ final definition.
Coming to Understand the Formal Definition of Limit

Amy: Is this going to be enough? . . . . What I’m having trouble with is just, is this [definition] specific enough to, like, to \( L \) being 4? I mean, isn’t this thing that we just said also true for 5 and 6 and 9.2, because . . . on that interval, \( f \) is continuous, and the maximum distances between \( y \)-values . . . and 10 are also decreasing. (Session 7)

This was a key moment in the teaching experiment. Amy recognized that the conditions of their definition would be met by values other than the limit value, 4, because for \( \text{any} \) number greater than 4, the maximum distances between that number and the \( y \)-values of the dampened sinusoidal function decrease as \( x \) increases. Apparently inspired by this observation, the students began to focus their defining efforts on eliminating all other potential limit candidates aside from the actual limit value.

The teacher–researcher attempted to support the students’ efforts by asking them to relax their standards momentarily and elaborate what it would mean for the function merely to be close to a proposed limit \( L \). He conjectured that this could lead the students (with support from the teacher–researcher) to think about progressively restricting the definition of closeness. In this way, he thought they might be able to adequately operationalize the troublesome notion of \textit{infinite closeness}.

I: You’re saying your definition . . . , you don’t want it to be such that someone could conclude . . . this limit could be anything other than 4.

---

8 Amy’s choice of the number 4 here is in reference to the dampened sinusoidal function shown in Figure 3 and Amy and Mike’s conversations about that function having a limit of 4 as \( x \to \infty \).

9 Amy recognized that the distances between 4 and the function values were oscillating, so she used the phrase \textit{maximum distance} for the maximum value of \(|f(x) - 4|\) on some interval \((x, \infty)\), noting that these maximum distances were decreasing as \( x \) approached infinity. She later noted that these are decreasing for any number greater than 4 (in fact, they are decreasing for any number).
Amy: Yeah.

I: . . . So let’s just say we want to maybe not show that 4 is the limit, but at least that this function gets close to 4. Because you guys were saying it’s got to get how close to 4 to be the limit?

Amy: Infinite.

I: Infinitely close, but let’s—ininitely is kind of tough, so let’s back off of infinitely for a second. Let’s say we just want to be close to 4. If we were able to show that this thing gets within 1 of 4, then that would keep the limit from being, say, 6 or 2.

Amy: Mm-hmm.

I: . . . Instead of having a limit of 4, let’s say, let’s describe what it would mean for this function to get within 1 of 4. How would you write that out? (Session 7)

This question seemed to be helpful, because Amy responded immediately by introducing the idea of bounding the limit value, setting the stage for iteratively tightening those bounds around this value.

Amy: Well, I feel like it would be useful to talk about it being, being bounded. . . . What if we were to say that there is some $y$-value that this function will never exceed, and there’s some $y$-value that it will never get bigger than and it will never get smaller than, you know? (Session 7)

After making this comment, Amy drew a vertical line from the dampened sinusoidal function down to the $x$-axis, indicating a point beyond which the function values would always be within the $y$-interval $(3, 5)$. She then, for the first time, drew horizontal bounds around $L$ on the $y$-axis at $y = 3$ and $y = 5$ to indicate the interval in which $f$ would fall within 1 of $L$ (Figure 4).10 Mike quickly made the observation that being within 1 of $L$ was much different from being infinitely close, saying that “the interval needs to be pretty much, like, 4. It needs to be from as close as you can below 4, as close as you can above 4, infinitely.” As he made this observation, Mike gestured with his hand, bringing his thumb and forefinger together apparently to indicate a tightening of the interval about 4. The teacher–researcher then asked Mike what the bounds around $y = 4$ would be for the function to lie within 1/2 of 4, in the hope of initiating a process of iteratively decreasing the size of the interval bounding the limit value. Mike drew horizontal lines bounding a 1-unit interval centered at 4. After drawing the new bounds, Mike noted that yet more limit candidates had been eliminated from consideration, saying, “We know the limit isn’t 5 anymore, because it’s bounded by 4 1/2 and 3 1/2.” Mike then observed that one could keep iterating this process of eliminating limit candidates by increasingly tightening the bounds around the limit value.

Mike: And we can keep doing that.

I: What do you mean we can keep doing that?

---

10 Note that the horizontal bounds Amy drew are not such that the function $f$ is always within 1 unit of $L = 4$ beyond the vertical line seen on the graph, which she had drawn previously. Amy’s spoken reasoning was such, however, that it was evident that this inaccuracy was merely an oversight on her part.
Mike: ... We can keep making our bounds closer and closer to 4, and the function will keep lying within those new bounds that we make. (Session 7)

This idea of iteratively increasing the required precision seemed to provide a starting point for attacking the problem of mathematizing the idea of infinite closeness. Mike’s statements roughly capture the idea of repeatedly tightening the interval around the proposed limit value and noting that by choosing large enough x-values, one can ensure that the function values are within that interval. This informal and dynamic idea (which we refer to as an iteratively closer perspective) is consistent with a formal definition of limit at infinity.

Although Amy also was approaching defining limit at infinity by bounding the function \( f \) around the limit \( L \) with increasingly tighter bounds, her perspective differed from Mike’s in an important, but subtle, way. Amy seems to have encapsulated the process that Mike had described, and developed a more powerful approach to mathematizing the idea of infinite closeness. Apparently as a result of reflecting on the process of iteratively defining closeness in an increasingly strict fashion, Amy explained that in order to prove that the limit (at infinity) of a function was 4, “you would say that you could make the bounds as close to 4 as you want. And you—as long as you take big enough x-values, you will find a point after which that function stays within those bounds.” We consider the formulation “as close to 4 as you want” to suggest that the existence of a limit can be established not by testing it for every definition of closeness, but rather by showing that it works for any arbitrarily chosen definition of closeness. This is confirmed by the fact that Amy then immediately introduced the term arbitrarily close in a written characterization of limit at infinity that contained the key components of the conventional definition of limit at infinity.

**Pair 1, Limit at Infinity:** It is possible to make bounds arbitrarily close to 4 and by taking large enough x-values we will find an interval \( (a, \infty) \) on which \( f(x) \) is within those bounds. (Session 7)

During Session 9, Amy explicitly described the relationship between the terms infinitely close and arbitrarily close.

Amy: The reason why I like the phrasing arbitrarily close better than infinitely close is because we can’t get infinitely close ‘cause it’s a kind of ... an abstract idea, but it’s
not something that we can practically do. Arbitrarily close means that like you pick a number as close as you, as you want, you know? And, and that’s something you can actually do and actually test, you know? Whereas you can’t test getting infinitely close to something.

I: So what would that testing look like for arbitrarily close? And what are you picturing in your mind?

Amy: Umm, arbitrarily close or arbitrarily small, you know like, i.e., I could actually take like the smallest real number that I can think of off the top of my head and throw it at it and test it and see whether the thing works, you know? Whereas like an infinitely small number I can’t come up with because that’s, you know, that’s an abstraction. (Session 9)

Amy’s final comment suggests that she viewed infinitely close as an abstraction that could be operationalized via the notion of arbitrarily close. At the outset of Session 8, the teacher–researcher encouraged Amy and Mike to incorporate the absolute value notation they had previously used to describe the distance between y-values and the limit candidate L. As they worked to include this notation in their definition, Amy developed the idea of using an arbitrarily small number to formulate the idea of arbitrarily close.

Amy: This L – y, absolute value of L – y is less than or equal to umm, a really, an arbitrarily tiny positive number. (Session 8)

Using Amy’s idea of an arbitrarily small number to capture the notion of arbitrarily close, and incorporating more precise notation, the students formulated a definition that captured the intended meaning of the conventional ε–N definition.

Pair 1, Limit at Infinity:

\[
\lim_{x \to \infty} f(x) = L
\]

provided for any arbitrarily small positive number λ, by taking sufficiently large values of x, we can find an interval \((a, \infty)\) such that for all \(x \in (a, \infty)\), \(|L - f(x)| \leq \lambda\). (Session 8, Final Definition)

In summary, by employing an arbitrary closeness perspective Amy encapsulated the process of progressively tightening what she referred to as the bounds around the limit value. This, in turn, made it possible for her to mathematize the infinite process of verifying that, for progressively smaller intervals about the limit value, one can ensure that the function values are within that interval by choosing a sufficiently large lower bound for the \(x\)-values. Although we view Mike’s iteratively closer perspective as productive for conceiving of a process one might use to validate a limit candidate, we suggest that Amy’s arbitrary closeness perspective is conducive to proving the existence of a limit.

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11 Later during Session 8, Amy noticed that the inclusion of the phrase by taking sufficiently large values of \(x\) was redundant, and noted that their definition could be streamlined to exclude this phrase.
The preceding discussion summarizes how the context of limit at infinity (as opposed to limit at a point) and the task of defining closeness (as opposed to infinite closeness) appeared to support Amy and Mike in resolving the challenge of operationalizing infinitely close. At the same time, the students also transitioned from an \( x \)-first perspective to a \( y \)-first perspective. This transition seems to have been initiated when the teacher–researcher asked them to describe what it would mean for a function to get within 1 unit of the limit value. Note that Amy’s first sketch of this situation (Figure 4) was drawn in a sequence that attended to the \( x \)-axis first. However, this set the stage for the students to progressively tighten the interval around the limit value and then find a point on the \( x \)-axis beyond which all the function values would lie within that interval. Thus, there was a shift in the students’ mathematical activity; they began to engage in an iterative \( y \)-first verification procedure. Ultimately, it was this \( y \)-first verification process that they were able to formalize (drawing on the notion of an arbitrarily small number) to create their formal definition of limit at infinity. It is worth noting, however, that even after successfully shifting to a \( y \)-first perspective, all 4 students remarked that such a perspective was counterintuitive.\(^{12}\)

The focal point of subsequent sessions returned to defining precisely the notion of limit at a point. As conjectured, Amy and Mike ultimately used their definition of limit at infinity as a template for defining limit at a point, arriving at a formulation consistent with the conventional \( \varepsilon – \delta \) definition.\(^{13}\)

**Pair 1, Definition 9:**

\[
\lim_{x \to a} f(x) = L
\]

provided that: given any arbitrarily small \( \# \lambda \), we can find an \((a \pm \theta)\) such that \(|L – f(x)| \leq \lambda\) for all \( x \) in that interval except possibly \( x = a \). (Session 9, Final Definition)

**Second teaching experiment.** In the preceding section, we described how defining closeness in the context of limit at infinity supported Amy and Mike in operationalizing the notion of infinite closeness and shifting to a \( y \)-first perspective. In the second experiment, we were interested to learn whether Chris and Jason could successfully reinvent the definition of limit at a point without first defining limit at infinity. We felt that defining what it means to be close had provided Amy and Mike a vehicle for operationalizing the notion of infinite closeness. Thus, we conjectured that Chris and Jason might realize the same cognitive shifts that Amy

\(^{12}\) For example, even as Chris and Jason were adopting a \( y \)-first perspective, Jason’s comments suggested that he found such a perspective counterintuitive:

I’m not saying that starting, you know, let this be your independent and then see what happens along the \( x \)-axis, I’m not saying there’s a problem with doing that. Just that it, it is foreign, uncomfortable, . . . and I’m not sure if it’s going to be in the realm of mathematical allowance.

\(^{13}\) Please see Swinyard (2011) for details of how Amy and Mike used their definition of limit at infinity to refine their definition of limit at a point.
and Mike experienced by characterizing what it means to be close in the context of limit at a point.

Midway through the second teaching experiment, Chris and Jason agreed that an unresolved issue was how they might characterize what it means for a function \( f \) to be infinitely close to a particular \( y \)-value, \( L \). In response, the teacher–researcher directed their attention to defining close.

I: We keep coming back to your original question, how close is close enough . . . let’s back off and try to just describe what close would mean . . . let’s say close means 10 units. So . . . how would you write out what it means for a function \( f(x) \), any function \( f(x) \), to be “close” under my definition to a particular predetermined value \( L \) for every \( x \) on the function?

In response, Chris and Jason drew the function shown in Figure 5, and introduced the idea of bounds on the \( y \)-axis, both above and below \( L \).

Jason: Let’s say, uh, this \( L, L = 10 \). Then we’ll have 15, and 5. There you go. In both directions, there you go. That is, for all \( x \)-values, always close to \( L \). ‘Cause it’s al-, well actually this would actually be 0 and 20. It’s always going to be within 10 units.

Chris: So you’re just bounding it.

This was a significant moment for Chris and Jason in the reinvention process, for it was the first time they had employed symmetric bounds on the \( y \)-axis about a particular \( y \)-value. Out of this discussion came their initial definition of close.

Chris: A function \( f(x) \) is close to \( L \) if and only if \( f(x) \) is within 10 units of \( L \).

Recall that the prompt for this task had been for Chris and Jason to define what it would mean for a function \( f \) to be close to a particular \( y \)-value, \( L \), for all values of \( x \). Unprompted, Jason recognized that their definition of close would need to be revised if they were to apply it to the limit concept: He reasoned that in the case of limit, they care only about closeness on an interval around the limiting point, \( x = a \).

Jason: With limit we’re talking about a very small portion of the domain and this close idea sounds like we’re trying to say, let’s be close for the whole function. And I don’t think that’s a requirement.

With the aim of shifting Chris and Jason’s attention back to defining limit, the teacher–researcher directed their attention to the step function shown in Figure 6 and asked them what the limit would be at the removable discontinuity located at the coordinate pair (3.5, 3). They agreed that the limit would be 3. The teacher–researcher next asked them when the function \( f \) would be within 2.5 units of the limit. As with the previous task, Jason added and subtracted the specified error
tolerance to the limit to form a $y$-interval around the proposed limit. Chris and Jason subsequently represented this $y$-interval with closed-interval notation, although they did so vertically, as seen in Figure 7, so as to denote that the interval was a $y$-interval, as opposed to an $x$-interval. This was significant because it was the first time they had used notation of any kind to denote bounds on the $y$-axis. Chris and Jason agreed that for the $x$-values in the interval $[1, 6)$ the function values would satisfy the specified definition of close. Chris and Jason then repeated this process as they described the set of $x$-values for which the function would be close under iteratively more restrictive definitions of close. This process represented both a way to work around the problematic notion of infinitely close and a shift to a $y$-first perspective.

$I$: I think we would agree that being within 2.5 of your limit, that’s not enough to mean that you have a limit . . . Let’s say that we wanted to be within, uh, 1.5, . . . or .5.

*Jason*: Then that shrinks [the $y$-interval] and then shrinks [the $x$-interval] . . .

*Chris*: So as [the $y$-interval] gets smaller you’re including less stuff.

$I$: Including less stuff, what do you mean by that?

*Chris*: Well like, when the range was 2 1/2, we went, we included 1, 2, 3, 4, 5, anything
that gave a value [between 1 and 5]. When it was 1 1/2, it would move up and we would not include anything that equates to 1 or 5 anymore. . . . We’re shrinking by picking smaller definitions of close.

Chris and Jason were ultimately unable to operationalize the notion of infinite closeness as Mike and Amy did during the first teaching experiment. However, by focusing on defining close, and subsequently discussing closeness in the context of the step function task, they did succeed in shifting to a y-first perspective.

Pair 2, Definition 9:

1. Come up with a guess, \( L \).
2. Determine a closeness interval \( L \pm z \) around your guess.
3. If: there exists an \( x_1 < a \) such that \( L + z > f(x) > L - z \) is true for all \( x \) between \( x_1 \) and \( a \) AND an \( x_2 > a \) such that \( L + z > f(x) > L - z \) is true for all \( x \) between \( x_2 \) and \( a \), then shrink your closeness interval and try again. If you can’t shrink your interval anymore, then \( L \) is your limit.

If not: then \( L \) is not your limit. (Session 10, Final Definition)

The first step of Chris and Jason’s final definition indicates their awareness that the purpose of the definition is to provide criteria for validating a predetermined limit candidate, \( L \), rather than to describe a process for finding a limit candidate. The second step reflects a y-first validation process that begins by choosing the equivalent of an \( \epsilon \)-neighborhood on the \( y \)-axis. Collectively, these two steps demonstrate a significant departure from their initial \( x \)-first definition. Although the third step in Chris and Jason’s definition does coordinate an \( x \)-interval containing \( a \) with the \( y \)-axis “closeness interval” \( (L \pm z) \), the phrase “If you can’t shrink your interval anymore” is problematic. Indeed, their requirement for concluding that \( L \) is the limit is impossible to meet, because there is no smallest positive real number. Chris and Jason’s iterative process, although similar to Mike’s (“We can keep making our bounds closer and closer to 4, and the function will keep lying within those new bounds that we make”), still contained vestiges of the idea of infinite closeness, in the form of a final jump to 0.

\textit{Jason}: If you can’t shrink your interval anymore, then that is capturing the \( z \) going to 0, right? . . . You know, it gets so far and then all of a sudden we have to say, okay, it jumps in a rocket and it shoots down to 0. You know? It’s not, it’s not approaching, it’s not going any more. We’re just saying, okay, well now take it to 0. Just go there.

Thus, Chris and Jason’s final definition still did not adequately address Jason’s previously stated concern: “So if there’s an infinite number of ‘closers’ between close and close enough, how can close enough even exist?”
CONCLUSIONS

The primary goal of this article is to build on existing research on students’ understanding of the formal definition of limit (e.g., Cottrill et al., 1996; Fernández, 2004; Tall & Vinner, 1981). In particular, we aim to refine the genetic decomposition of limit proposed by Cottrill et al. (1996) based on our retrospective analysis of two teaching experiments designed to investigate students’ reasoning about limit in the context of reinventing a formal definition. Each teaching experiment began with the students generating examples based on their concept images of limit. Guided by these examples, the students attempted to precisely define the limit concept. Two central challenges arose: (a) students relied on an $x$-first perspective and were reluctant to employ a $y$-first perspective; and (b) students struggled to operationalize what it means to be infinitely close to a point. To address these two challenges, we employed instructional tasks designed to focus the students’ attention on describing what it means to be close to a limit value. The goal was to encourage the students to develop a process of defining closeness in an iteratively restrictive fashion. As illustrated previously, Amy and Mike were able to develop such a process, and Amy was able to encapsulate this process via the notion of arbitrary closeness. Further, this notion of arbitrary closeness became part of a $y$-first limit-validating process that Amy and Mike were able to formulate in a coherent definition logically equivalent to the conventional $\varepsilon$–$\delta$ definition.

Based on our analysis of the students’ reasoning, we are able to build on the Cottrill et al. (1996) framework by addressing the question of how students shift from the $x$-first process described in Step 3 (as $x$ gets closer to $a$, $f(x)$ gets closer to $L$) to the more formally expressed $y$-first process (for every $\varepsilon > 0$, there exists a $\delta > 0$, such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$) described in the formal definition. We propose that a genetic decomposition of limit could include the first three steps of the Cottrill et al. framework, along with the following three steps that address the process of coming to understand the formal definition:

4. Constructing a mental process in which one tests whether a given candidate is a limit by:
   a. Choosing a measure of closeness to the limit value $L$ along the $y$-axis;
   b. Determining whether there is an interval around the point at which one is taking the limit (i.e., $a$) for which every function value aside from the one at that point is close enough to $L$; and
   c. Repeating this for smaller and smaller measures of closeness.
5. Associating the existence of a limit with the ability to continue (theoretically) this process forever without failing to produce the desired interval about $a$, or equivalently with the observation that there is no point at which it will be impossible to find such an interval.
6. Encapsulating this process via the concept of arbitrary closeness. This involves realizing that one can establish that the process in Step 4 will work for every
possible measure of closeness by proving that it will work for an arbitrary measure of closeness.

Step 4 of our proposed genetic decomposition characterizes the necessary shift to a \( y \)-first limit-validation process. This process involves iteratively defining an error bound about the limit candidate \( L \) (along the \( y \)-axis) and then determining whether an interval can be found about \( a \) (along the \( x \)-axis) for which every function value (except possibly at \( x = a \)) lies within the error bound of \( L \). This is exemplified by Chris and Jason’s validation process in which one is instructed to repeatedly “shrink your closeness interval and try again.” (Both pairs of students acknowledged that iteratively defining an error bound about the limit candidate \( L \) resulted in eliminating extraneous limit candidates.) What is missing in this conception is the ability to “step back” from this process and associate it with the existence of a limit.

Step 5 of the decomposition emphasizes the mental action of associating the infinite process described in Step 4 with a criterion for validating a limit candidate. If this process can be repeated indefinitely with success, then \( L \) is the limit. This is exemplified by Mike’s observation (in the case of a limit at infinity) that if the limit is 4, “we can keep doing that. . . . We can keep making our bounds closer and closer to 4, and the function will keep lying within those new bounds that we make.” This is an important step, because it provides a criterion that can be used (at least in informal arguments) to prove a limit candidate is (or is not) a limit.

Step 6 represents an encapsulation of this validation process and is exemplified by Amy’s use of arbitrarily small numbers to operationalize the idea of infinitely close. This step is crucial for the development of a robust formal understanding of limit, because it includes an understanding of the formal tools that make it possible to prove general statements about limits (including the pervasive “let epsilon be greater than 0” proof format).

Although this theoretical contribution is the result of our work with just two pairs of students in the context of reinvention, it is reasonable to expect that it could be more broadly relevant. These three additional steps represent significant mathematical differences between the formal definition of limit and the informal conceptions typical of 1st-year calculus students (Cottrill et al., 1996; Tall & Vinner, 1981; Williams, 1991). Thus, it seems likely that students taking their first analysis course would need to develop the understandings elucidated by Steps 4, 5, and 6 along the way to learning the formal definition of limit.

Further, our analysis of the students’ reasoning suggests possible approaches to supporting students in developing these understandings. For example, limit at infinity seems to be a productive context for shifting students’ attention to the \( y \)-axis. Additionally, the activity of repeatedly choosing smaller error tolerances to exclude extraneous limit candidates seems to support the development of an infinite process involving iteratively restricting the definition of close. Together these two instructional strategies appear to lay the stage for developing a formal definition of limit.

Further research needs to be conducted to investigate the extent to which our
findings are generalizable beyond the context of reinvention. For example, in a typical first analysis course, students are generally presented with a formal definition of limit instead of being asked to reinvent one. Thus, a further area of research would be to investigate the extent to which the reasoning described in Steps 4, 5, and 6 of our proposed genetic decomposition are an important part of making sense of such a definition. Additionally, further instructional design research (including whole-class teaching experiments) would be necessary to develop and test instructional materials based on our findings.

REFERENCES


Coming to Understand the Formal Definition of Limit


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Accepted December 5, 2011