

## Chapter 8: Nonhomogeneous Problems

### Heat flow with sources and nonhomogeneous boundary conditions

We consider first the heat equation without sources and constant nonhomogeneous boundary conditions.

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad (1)$$

$$u(0, t) = A \quad (2)$$

$$u(L, t) = B \quad (3)$$

$$u(x, 0) = f(x) \quad (4)$$

In this case the method of separation of variables does not work since the boundary conditions are nonhomogeneous.

To solve the problem we use the following approach: first we find the equilibrium temperature  $u_E(x)$  by solving the problem

$$\frac{d^2 u_E}{dx^2} = 0 \quad (5)$$

$$u_E(0) = A \quad (6)$$

$$u_E(L) = B \quad (7)$$

The solution is

$$u_E(x) = A + \frac{B - A}{L}x$$

Next we introduce a new function  $v(x, t)$  that measures the displacement of the temperature  $u(x, t)$  from the equilibrium temperature  $u_E(x)$

$$v(x, t) = u(x, t) - u_E(x)$$

Then  $v(x, t)$  is the solution of the homogeneous problem

$$\frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2} \quad (8)$$

$$v(0, t) = 0 \quad (9)$$

$$v(L, t) = 0 \quad (10)$$

$$v(x, 0) = f(x) - u_E(x) \quad (11)$$

and is given by

$$v(x, t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2 t}$$

and from (11) we determine the coefficients  $a_n$  as the Fourier sine series coefficients of  $f(x) - u_E(x)$

$$a_n = \frac{2}{L} \int_0^L [f(x) - u_E(x)] \sin \frac{n\pi x}{L} dx$$

Then, the solution of the problem (1-4) is

$$u(x, t) = u_E(x) + v(x, t) = A + \frac{B - A}{L}x + \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2 t}$$

where the coefficients  $a_n$  are evaluated according to the formulae above. Notice that as time goes to infinity,  $u(x, t)$  approaches the equilibrium temperature  $u_E(x)$ .

The approach described above will work also for the nonhomogeneous problem with time independent (steady) sources of thermal energy

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + Q(x) \quad (12)$$

subject to (2), (3), (4) if an equilibrium temperature exists.

#### TIME-DEPENDENT NONHOMOGENEOUS TERMS

Next we consider the following nonhomogeneous problem

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + Q(x, t) \quad (13)$$

$$u(0, t) = A(t) \quad (14)$$

$$u(L, t) = B(t) \quad (15)$$

$$u(x, 0) = f(x) \quad (16)$$

In this case the sources of thermal energy are time and space dependent, and the boundary values at the ends may vary in time.

We will solve this problem in two steps:

1. In the first step we look for a function  $r(x, t)$  such that  $r(0, t) = A(t)$  and  $r(L, t) = B(t)$ . Moreover we want  $r(x, t)$  to have a simple form, for example

$$r(x, t) = A(t) + \frac{x}{L}[B(t) - A(t)] \quad (17)$$

2. In the second step we consider the difference

$$v(x, t) = u(x, t) - r(x, t) \Leftrightarrow u(x, t) = v(x, t) + r(x, t) \quad (18)$$

Then,  $v(x, t)$  is the solution of the problem

$$\frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2} + \left[ Q(x, t) - \frac{\partial r}{\partial t} + k \frac{\partial^2 r}{\partial x^2} \right] \equiv k \frac{\partial^2 v}{\partial x^2} + \bar{Q}(x, t) \quad (19)$$

with homogeneous boundary conditions

$$v(0, t) = 0 \quad (20)$$

$$v(L, t) = 0 \quad (21)$$

and the initial condition

$$v(x, 0) = u(x, 0) - r(x, 0) = f(x) - A(0) - \frac{x}{L}[B(0) - A(0)] \equiv g(x) \quad (22)$$

The problem is reduced now to solving (19-22). To solve this problem we use the *method of eigenfunction expansion*.

We consider the homogeneous problem

$$\frac{\partial w}{\partial t} = k \frac{\partial^2 w}{\partial x^2} \quad (23)$$

$$w(0, t) = 0 \quad (24)$$

$$w(L, t) = 0 \quad (25)$$

By using separation of variables we obtain the eigenvalue problem

$$\frac{d^2 \Phi}{dx^2} + \lambda \Phi = 0$$

$$\Phi(0) = 0$$

$$\Phi(L) = 0$$

For this problem the eigenvalues are  $\lambda_n = \left(\frac{n\pi}{L}\right)^2$  and the corresponding eigenfunctions are  $\Phi_n(x) = \sin \frac{n\pi x}{L}$ . To maintain generality, we do not use the explicit form of the eigenfunctions. We express the unknown solution  $v(x, t)$  as a generalized Fourier series of eigenfunctions with time dependent coefficients

$$v(x, t) = \sum_{n=1}^{\infty} a_n(t) \Phi_n(x) \quad (26)$$

Assuming that term-by-term differentiation is justified, after replacing (26) in (19) we obtain

$$\sum_{n=1}^{\infty} \frac{da_n(t)}{dt} \Phi_n(x) = k \sum_{n=1}^{\infty} a_n(t) \frac{d^2 \Phi_n(x)}{dx^2} + \bar{Q}(x, t) = -k \sum_{n=1}^{\infty} a_n(t) \lambda_n \Phi_n(x) + \bar{Q}(x, t) \quad (27)$$

To obtain the last relation in (27) we used the eigenfunction property  $\frac{d^2 \Phi_n}{dx^2} = -\lambda_n \Phi_n$ .

Next we also expand  $\bar{Q}(x, t)$  as a generalized Fourier series of eigenfunctions with time dependent coefficients

$$\bar{Q}(x, t) = \sum_{n=1}^{\infty} \bar{q}_n(t) \Phi_n(x) \quad (28)$$

where the generalized Fourier coefficients are given by

$$\bar{q}_n(t) = \frac{\int_0^L \bar{Q}(x, t) \Phi_n(x) dx}{\int_0^L \Phi_n^2(x) dx} \quad (29)$$

After replacing (28) in (27) and identifying the coefficients, we obtain the following differential equations for the coefficients

$$\frac{da_n}{dt} + \lambda_n k a_n = \bar{q}_n(t), \quad n = 1, 2, \dots \quad (30)$$

To obtain the initial conditions for the differential equations (30) we use the initial condition (22). Then

$$g(x) = \sum_{n=1}^{\infty} a_n(0) \Phi_n(x)$$

such that

$$a_n(0) = \frac{\int_0^L g(x)\Phi_n(x)dx}{\int_0^L \Phi_n^2(x)dx}, \quad n = 1, 2, \dots \quad (31)$$

**Q:** Show that the solution of the initial-value problem (30)-(31) is given by

$$a_n(t) = a_n(0)e^{-\lambda_n kt} + e^{-\lambda_n kt} \int_0^t \bar{q}_n(\tau)e^{\lambda_n k\tau} d\tau \quad (32)$$

To conclude, the solution of the nonhomogeneous problem (13-16) is

$$u(x, t) = v(x, t) + r(x, t) \quad (33)$$

where  $r(x, t)$  is given by (17) and  $v(x, t)$  is given by (26) with the coefficients  $a_n(t)$  evaluated according to (32).

**Q:** Solve the nonhomogeneous problem

$$\begin{aligned} u_t &= u_{xx} + e^{-t} \sin(3x), & 0 < x < \pi \\ u(0, t) &= 0, & u(\pi, t) = 1, & t > 0 \\ u(x, 0) &= f(x), & 0 \leq x \leq \pi \end{aligned}$$

### Method of eigenfunction expansion using Green's formula

We consider the heat equation with sources and nonhomogeneous time dependent boundary conditions.

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + Q(x, t) \quad (34)$$

$$u(0, t) = A(t) \quad (35)$$

$$u(L, t) = B(t) \quad (36)$$

$$u(x, 0) = f(x) \quad (37)$$

Previously we learned how to solve this problem in two steps:

1. In the first step we find a function  $r(x, t)$  such that  $r(0, t) = A(t)$ ,  $r(L, t) = B(t)$ . Then the difference  $v(x, t) = u(x, t) - r(x, t)$  will satisfy a problem with homogeneous boundary conditions.
2. In the second step the solution  $v(x, t)$  is obtained using the method of eigenfunction expansion, then  $u(x, t) = v(x, t) + r(x, t)$ .

Next we show how the method of eigenfunction expansion may be applied directly to solve the problem (34-37). Special attention must be paid to the fact that term-by-term differentiation of the Fourier series is not justified when the boundary conditions are nonhomogeneous.

To begin, we consider the homogeneous problem

$$\frac{d^2 \Phi}{dx^2} + \lambda_n \Phi = 0 \quad (38)$$

$$\Phi(0) = 0 \quad (39)$$

$$\Phi(L) = 0 \quad (40)$$

For this problem the eigenvalues are  $\lambda_n = (n\pi/L)^2$  and the corresponding eigenfunctions are  $\Phi_n = \sin \frac{n\pi x}{L}$ . Since any piecewise smooth function may be expanded as a series of these eigenfunctions, we consider

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t) \Phi_n(x) \quad (41)$$

The (generalized) Fourier coefficients  $b_n(t)$  may be expressed as

$$b_n(t) = \frac{\int_0^L u \Phi_n dx}{\int_0^L \Phi_n^2 dx} \quad (42)$$

Notice that at the points where the (periodic) extension of  $u(x, t)$  is not continuous (e.g.  $x = 0$  and  $x = L$ ) the " = " sign in (41) must be replaced by "  $\sim$ ". Term by term differentiation with respect to  $x$  is not justified since  $u(x, t)$  satisfies nonhomogeneous boundary conditions. We can only consider term-by-term differentiation of (41) with respect to time

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{db_n}{dt} \Phi_n(x) \quad (43)$$

After replacing (43) in (34) we obtain

$$\sum_{n=1}^{\infty} \frac{db_n}{dt} \Phi_n(x) = k \frac{\partial^2 u}{\partial x^2} + Q(x, t)$$

such that, using the orthogonality of the eigenfunctions, it follows that

$$\frac{db_n}{dt} = \frac{\int_0^L \left[ k \frac{\partial^2 u}{\partial x^2} + Q(x, t) \right] \Phi_n(x) dx}{\int_0^L \Phi_n^2(x) dx} \quad (44)$$

We want to obtain a first-order differential equation for the coefficients  $b_n(t)$ . We notice that if we consider the (generalized) Fourier series of the source terms

$$Q(x, t) = \sum_{n=1}^{\infty} q_n(t) \Phi_n(x)$$

then the coefficients  $q_n(t)$  are given by

$$q_n(t) = \frac{\int_0^L Q(x, t) \Phi_n(x) dx}{\int_0^L \Phi_n^2(x) dx} \quad (45)$$

Using (45) we may write (44) as

$$\frac{db_n}{dt} = q_n(t) + k \frac{\int_0^L \frac{\partial^2 u}{\partial x^2} \Phi_n(x) dx}{\int_0^L \Phi_n^2(x) dx} \quad (46)$$

To express the last term in the right hand side of (46) in terms of  $b_n(t)$  we use the following property known as *Green's formula*

$$\int_0^L \left( u \frac{\partial^2 v}{\partial x^2} - v \frac{\partial^2 u}{\partial x^2} \right) dx = \left( u \frac{\partial v}{\partial x} - v \frac{\partial u}{\partial x} \right) \Big|_0^L \quad (47)$$

**Q:** Prove that (47) is valid for any functions  $u, v$  that are twice continuously differentiable.

If we let  $v = \Phi_n(x)$  in (47) then we obtain

$$\int_0^L \Phi_n(x) \frac{\partial^2 u}{\partial x^2} dx = \int_0^L u \frac{d^2 \Phi_n}{dx^2} dx - \left( u \frac{d\Phi_n}{dx} - \Phi_n \frac{\partial u}{\partial x} \right) \Big|_0^L \quad (48)$$

Since the eigenfunction  $\Phi_n$  satisfies (38-40) with  $\lambda = \lambda_n$ , from (48) we obtain

$$\int_0^L \Phi_n(x) \frac{\partial^2 u}{\partial x^2} dx = -\lambda_n \int_0^L u \Phi_n dx - u \frac{d\Phi_n}{dx} \Big|_0^L = -\lambda_n \int_0^L u \Phi_n dx - \frac{n\pi}{L} [B(t)(-1)^n - A(t)] \quad (49)$$

After replacing (49) and (42) in (46) we obtain a first-order linear differential equation for the Fourier coefficients

$$\frac{db_n}{dt} + k\lambda_n b_n = q_n(t) + \frac{k \frac{n\pi}{L} [A(t) - (-1)^n B(t)]}{\int_0^L \Phi_n^2(x) dx} \quad (50)$$

The required initial condition for (50) follows from the given initial condition (37)

$$f(x) = \sum_{n=1}^{\infty} b_n(0) \Phi_n(x)$$

$$b_n(0) = \frac{\int_0^L f(x) \Phi_n(x) dx}{\int_0^L \Phi_n^2(x) dx} \quad (51)$$

Therefore, the solution  $u(x, t)$  of (34-37) is given by

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t) \Phi_n(x) \quad (52)$$

where the coefficients are obtained by solving the initial-value problem (50)-(51).

### Forced vibrating string and resonance

We consider vibrations of a string subject to an external force  $Q(x, t)$ . The model problem is given by the PDE

$$u_{tt} = c^2 u_{xx} + Q(x, t), \quad 0 < x < L, \quad t > 0 \quad (53)$$

with homogeneous boundary conditions,

$$u(0, t) = 0 \quad (54)$$

$$u(L, t) = 0 \quad (55)$$

and initial conditions

$$u(x, 0) = f(x), \quad 0 < x < L \quad (56)$$

$$\frac{\partial u}{\partial t}(x, 0) = g(x), \quad 0 < x < L \quad (57)$$

We search for the solution  $u(x, t)$  as a series of x-eigenfunctions with time dependent coefficients

$$u(x, t) = \sum_{n=1}^{\infty} a_n(t) \phi_n(x) \quad (58)$$

where  $\phi_n(x) = \sin(n\pi x/L)$ ,  $\lambda_n = (n\pi/L)^2$ . We also consider the Fourier series of the forcing term,

$$Q(x, t) = \sum_{n=1}^{\infty} q_n(t) \phi_n(x) \quad (59)$$

where the time dependent coefficients  $q_n(t)$  are computed according to

$$q_n(t) = \frac{\int_0^L Q(x, t) \phi_n(x) dx}{\int_0^L \phi_n^2(x) dx} = \frac{2}{L} \int_0^L Q(x, t) \sin\left(\frac{n\pi x}{L}\right) dx \quad (60)$$

After replacing (58) and (59) into the PDE (53) and using the fact that  $\phi_n'' = -\lambda_n \phi_n$ , we obtain:

$$\sum_{n=1}^{\infty} a_n''(t) \phi_n(x) = -c^2 \sum_{n=1}^{\infty} \lambda_n a_n(t) \phi_n(x) + \sum_{n=1}^{\infty} q_n(t) \phi_n(x) \quad (61)$$

Identification of the coefficients in the series above implies that the coefficients  $a_n(t)$  will satisfy the ODEs

$$a_n''(t) + c^2 \lambda_n a_n(t) = q_n(t), \quad n = 1, 2, \dots \quad (62)$$

To obtain initial conditions for these ODEs we use (58) in (56-57) such that  $a_n(0)$  must be the Fourier coefficients of  $f(x)$  and  $a_n'(0)$  must be Fourier coefficients of  $g(x)$

$$a_n(0) = \frac{\int_0^L f(x) \phi_n(x) dx}{\int_0^L \phi_n^2(x) dx}; \quad a_n'(0) = \frac{\int_0^L g(x) \phi_n(x) dx}{\int_0^L \phi_n^2(x) dx} \quad (63)$$

The second order ODEs (62) has the general solution as the sum of the general solution to the homogeneous equation and a particular solution, call it  $a_n^p(t)$ , to the nonhomogeneous equation

$$a_n(t) = c_1 \cos(c\sqrt{\lambda_n} t) + c_2 \sin(c\sqrt{\lambda_n} t) + a_n^p(t)$$

The constants  $c_1, c_2$  above are determined from the initial conditions (63). A particular solution  $a_n^p(t)$  may be obtained using the method of variation of parameters.

As a particular case, we consider the vibrations of a string subject to a periodic forcing  $Q(x, t) = q(x) \cos(\omega t)$ , where  $\omega > 0$  is a constant. In this case, writing the forcing term as a series of eigenfunctions we get

$$q(x) \cos(\omega t) = \sum_{n=1}^{\infty} q_n \phi_n(x) \cos(\omega t), \quad q_n = \frac{2}{L} \int_0^L q(x) \sin(n\pi x/L) dx \quad (64)$$

and the differential equations (62) for the coefficients are written

$$a_n''(t) + c^2 \lambda_n a_n(t) = q_n \cos(\omega t) \quad (65)$$

**Q:** Show that the general solution to (65) is

$$a_n(t) = c_1 \cos(c\sqrt{\lambda_n} t) + c_2 \sin(c\sqrt{\lambda_n} t) + \frac{q_n \cos(\omega t)}{c^2 \lambda_n - \omega^2} \quad (66)$$

where  $c_1, c_2$  are arbitrary constants, provided that *the forcing frequency  $\omega$  is different from the natural frequency  $c\lambda_n$  :  $c^2 \lambda_n - \omega^2 \neq 0$ .*

**Q:** Show that if **the forcing frequency  $\omega$  is the same as one of the natural frequencies  $c\lambda_n$**  then the general solution to (65) is

$$a_n(t) = c_1 \cos(c\sqrt{\lambda_n} t) + c_2 \sin(c\sqrt{\lambda_n} t) + \frac{q_n}{2\omega} t \sin(\omega t) \quad (67)$$

and **resonance** occurs: the amplitude of the natural modes corresponding to the forcing frequency will grow in time without a bound.

**Q:** Solve the problem (53- 57) in  $0 \leq x \leq \pi$  if  $c^2 = 1$  and

1.  $Q(x, t) = \sin x \sin t$

2.  $Q(x, t) = \sin x \cos 2t$

**Q:** Repeat the problem above when the Dirichlet boundary conditions (54-55) are replaced by the "free end" boundary conditions

$$u_x(0, t) = 0 \tag{68}$$

$$u_x(L, t) = 0 \tag{69}$$



## Section 8.6: Poisson's Equation

We consider a two dimensional domain  $\Omega$  with boundary  $S$  and the *Poisson's equation*

$$\nabla^2 u = Q, \quad \text{in } \Omega \quad (1)$$

subject to nonhomogeneous boundary conditions

$$u = \alpha, \quad \text{on } S \quad (2)$$

The function  $u$  (equilibrium temperature), the source terms  $Q$ , and the boundary values  $\alpha$  are time independent:  $u = u(x, y)$ ,  $Q = Q(x, y)$ ,  $\alpha = \alpha(x, y)$ .

Our first approach to find the solution is to split problem (1-2) in two problems that are easier to solve:

1. a nonhomogeneous PDE

$$\nabla^2 u_1 = Q, \quad \text{in } \Omega \quad (3)$$

subject to the homogeneous boundary condition

$$u_1 = 0, \quad \text{on } S \quad (4)$$

2. a homogeneous (Laplace) PDE

$$\nabla^2 u_2 = 0, \quad \text{in } \Omega \quad (5)$$

subject to the nonhomogeneous boundary condition

$$u_2 = \alpha, \quad \text{on } S \quad (6)$$

If we are able to solve these problems, using the linearity we can easily show that

$$u = u_1 + u_2 \quad (7)$$

is the solution of the nonhomogeneous problem (1-2).

We have already studied the nonhomogeneous problem (Laplace) (5)-(6). If  $\Omega$  has a simple geometrical shape (e.g. rectangle, disk) the solution may be obtained using separation of variables. Next we focus our attention on solving problem (3)-(4) on a rectangle:  $0 \leq x \leq L, 0 \leq y \leq H$ .

Using the method of eigenfunction expansion we expand  $u_1(x, y)$  in a series of  $x$ -eigenfunctions

$$u_1(x, y) = \sum_{n=1}^{\infty} b_n(y) \sin \frac{n\pi x}{L} \quad (8)$$

where the Fourier sine series coefficients  $b_n$  are functions of  $y$ .

*Remark: alternatively, we may expand  $u_1(x, y)$  in a series of  $y$ -eigenfunctions*

$$u_1(x, y) = \sum_{n=1}^{\infty} b_n(x) \sin \frac{n\pi y}{H} \quad (9)$$

*We are free to make this choice since the boundary are homogeneous in both  $x$  and  $y$  directions.*

By differentiating (8) twice with respect to  $y$  then replacing in (3) we obtain

$$\sum_{n=1}^{\infty} \frac{d^2 b_n}{dy^2} \sin \frac{n\pi x}{L} + \frac{\partial^2 u_1}{\partial x^2} = Q(x, y) \quad (10)$$

Term-by-term differentiation of the Fourier sine series (8) with respect to  $x$  is also justified since  $u_1(0, y) \equiv 0, u_1(L, y) \equiv 0$ . By differentiating term-by-term (8) twice with respect to  $x$  we obtain

$$\frac{\partial^2 u_1}{\partial x^2} = - \sum_{n=1}^{\infty} \left( \frac{n\pi}{L} \right)^2 b_n(y) \sin \frac{n\pi x}{L}$$

such that we may write (10) as

$$\sum_{n=1}^{\infty} \left[ \frac{d^2 b_n}{dy^2} - \left( \frac{n\pi}{L} \right)^2 b_n \right] \sin \frac{n\pi x}{L} = Q(x, y) \quad (11)$$

We may also expand the source terms as a Fourier sine series

$$Q(x, y) = \sum_{n=1}^{\infty} q_n(y) \sin \frac{n\pi x}{L} \quad (12)$$

where the coefficients  $q_n(y)$  are

$$q_n(y) = \frac{2}{L} \int_0^L Q(x, y) \sin \frac{n\pi x}{L} dx \quad (13)$$

After replacing (12) in (11) and identifying the coefficients we obtain a second-order nonhomogeneous differential equation for  $b_n(y)$ :

$$\frac{d^2 b_n}{dy^2} - \left( \frac{n\pi}{L} \right)^2 b_n = q_n(y) \quad (14)$$

Using the boundary conditions  $u_1(x, 0) \equiv 0, u_1(x, H) \equiv 0$  in (8) we obtain the boundary conditions for the coefficients

$$b_n(0) = 0 \quad (15)$$

$$b_n(H) = 0 \quad (16)$$

Therefore, the coefficients are obtained by solving the nonhomogeneous boundary value problem (14), (15), (16). The general solution of the homogeneous equation

$$\frac{d^2 b_n^o}{dy^2} - \left( \frac{n\pi}{L} \right)^2 b_n^o = 0 \quad (17)$$

is

$$b_n^o(y) = a_1 e^{\frac{n\pi y}{L}} + a_2 e^{-\frac{n\pi y}{L}} \quad (18)$$

which may be written in the equivalent form

$$b_n^o(y) = c_1 \sinh \frac{n\pi}{L}(y - H) + c_2 \cosh \frac{n\pi}{L}(y - H) \quad (19)$$

The general solution of the nonhomogeneous equation (14) is obtained as

$$b_n(y) = b_n^o(y) + b_n^p(y) = c_1 \sinh \frac{n\pi}{L}(y - H) + c_2 \cosh \frac{n\pi}{L}(y - H) + b_n^p(y) \quad (20)$$

where  $b_n^p(y)$  is a particular solution of (14) (that means any function that satisfies (14)). To find the solution of the boundary value problem (14),(15-16) we determine the constants  $c_1$  and  $c_2$  in (20) using the boundary conditions (15-16).

A particular solution may be found using the method of variations of parameters. This approach is general, but is not always trivial and may require extensive computations. In some special situations (if  $q_n(y)$  has a simple form ) a particular solution may be found by searching for  $b_n^p(y)$  of the same form as  $q_n(y)$ .

**Q:** Solve the problem (14),(15-16) if

- a.  $q_n(y) = 1$
- b.  $q_n(y) = y^2$
- c.  $q_n(y) = e^{-y}$
- d.  $q_n(y) = \cos y$

### Nonhomogeneous boundary conditions (any geometry)

Next we show how the method of eigenfunction expansion may be used to solve the Poisson's equation

$$\nabla^2 u = Q, \quad \text{in } \Omega \quad (21)$$

with nonhomogeneous boundary conditions

$$u = \alpha, \quad \text{on } S \quad (22)$$

where  $\Omega$  is an arbitrary (2D or 3D) domain with boundary  $S$ . We assume that the eigenvalue problem  $\nabla^2 \Phi = -\lambda \Phi$  with  $\Phi = 0$  on  $S$  has been solved such that the eigenvalues  $\lambda_i$  and the corresponding eigenfunctions  $\Phi_i$  are assumed to be known.

We express the function  $u(x, y)$  as a series of eigenfunctions

$$u = \sum_i b_i \Phi_i \quad (23)$$

and search for the coefficients  $b_i$ . Term-by-term differentiation of (23) is not justified since  $u$  satisfies nonhomogeneous boundary conditions. Using the orthogonality of the eigenfunctions from (23) we obtain

$$b_i = \frac{\int_{\Omega} u \Phi_i d\Omega}{\int_{\Omega} \Phi_i^2 d\Omega} = -\frac{1}{\lambda_i} \frac{\int_{\Omega} u \nabla^2 \Phi_i d\Omega}{\int_{\Omega} \Phi_i^2 d\Omega} \quad (24)$$

The last relation in (24) is obtained using  $\nabla^2 \Phi_i = -\lambda_i \Phi_i$ .

**Q:** Prove that for any functions  $u$  and  $v$  that are twice continuously differentiable the following *Green's formula* holds

$$\int_{\Omega} (u \nabla^2 v - v \nabla^2 u) d\Omega = \int_S (u \nabla v - v \nabla u) \cdot \mathbf{n} ds \quad (25)$$

where  $\mathbf{n}$  is the unit outward normal vector to the boundary.

**Q:** Show that the orthogonality of the eigenfunctions follows directly from the Green's formula (25). Using (25) with  $v = \Phi_i$  we obtain

$$\int_{\Omega} u \nabla^2 \Phi_i d\Omega = \int_{\Omega} \Phi_i \nabla^2 u d\Omega + \int_S (u \nabla \Phi_i - \Phi_i \nabla u) \cdot \mathbf{n} ds \quad (26)$$

Since  $\nabla^2 u = Q$  in  $\Omega$  and  $u = \alpha$ ,  $\Phi_i = 0$  on  $S$ , (26) is written

$$\int_{\Omega} u \nabla^2 \Phi_i d\Omega = \int_{\Omega} Q \Phi_i d\Omega + \int_S \alpha \nabla \Phi_i \cdot \mathbf{n} ds$$

such that from (24) we may express the coefficients  $b_i$  as

$$b_i = -\frac{1}{\lambda_i} \frac{\int_{\Omega} \Phi_i Q d\Omega + \int_S \alpha \nabla \Phi_i \cdot \mathbf{n} ds}{\int_{\Omega} \Phi_i^2 d\Omega} \quad (27)$$