Mth 251: Calculus I

Midterm exam: Solutions

PRINT your name: A STUDENT

The exam has 6 problems and each problem is worth 10 points.

In order to receive full credit you must show all work.

Good luck!
**Problem 1.** Evaluate the limit

\[
\lim_{x \to 2} \frac{x^2 - 3x + 2}{2x^2 - 8}
\]

**Solution** Notice that we have an indeterminate limit case of \( \frac{0}{0} \).

With a few algebraic manipulations, we may simplify this limit as follows

\[
\frac{x^2 - 3x + 2}{2x^2 - 8} = \frac{(x - 2)(x - 1)}{2(x^2 - 4)} = \frac{(x - 2)(x - 1)}{2(x - 2)(x + 2)} = \frac{x - 1}{2(x + 2)}
\]

Thus,

\[
\lim_{x \to 2} \frac{x^2 - 3x + 2}{2x^2 - 8} = \lim_{x \to 2} \frac{x - 1}{2(x + 2)} = \frac{2 - 1}{2(2 + 2)} = \frac{1}{8}
\]
**Problem 2.** Evaluate the limit
\[
\lim_{x \to 0} \frac{1 - \cos x}{\sin^2 x}
\]

**Solution** Notice that we have an indeterminate limit case of \(\frac{0}{0}\), since \(\cos 0 = 1\) and \(\sin 0 = 0\)

We may use the identity
\[
\sin^2 x + \cos^2 x = 1
\]

to obtain
\[
\sin^2 x = 1 - \cos^2 x
\]
and write the limit as
\[
\lim_{x \to 0} \frac{1 - \cos x}{\sin^2 x} = \lim_{x \to 0} \frac{1 - \cos x}{1 - \cos^2 x} = \lim_{x \to 0} \frac{1 - \cos x}{(1 - \cos x)(1 + \cos x)} = \lim_{x \to 0} \frac{1}{1 + \cos x} = \frac{1}{2}
\]
Problem 3. Evaluate the limit

\[ \lim_{x \to \infty} (\sqrt{x+1} - \sqrt{x+2}) \]

Solution Notice that we have an indeterminate limit case \( \infty - \infty \).

We can resolve it with the help of the conjugate expression, as follows

\[ (\sqrt{x+1} - \sqrt{x+2}) = \frac{(\sqrt{x+1} - \sqrt{x+2})(\sqrt{x+1} + \sqrt{x+2})}{(\sqrt{x+1} + \sqrt{x+2})} \]

\[ = \frac{(x+1) - (x+2)}{(\sqrt{x+1} + \sqrt{x+2})} = \frac{-1}{(\sqrt{x+1} + \sqrt{x+2})} \]

Thus,

\[ \lim_{x \to \infty} (\sqrt{x+1} - \sqrt{x+2}) = \lim_{x \to \infty} \frac{-1}{\sqrt{x+1} + \sqrt{x+2}} = 0 \]

since the denominator increases without bound to \( \infty \) as \( x \to \infty \).
**Problem 4.** Use the Intermediate Value Theorem to find an interval of length 1 containing a root of the equation

\[ 2^x - x^3 = 0 \]

**Solution** Since

\[ f(x) = 2^x - x^3 \]

is a continuous function, to apply the IVT all we need is to find an interval \((a, b)\) such that \(b - a = 1\) (that is of length 1) and

\[ f(a) \cdot f(b) < 0 \]

Notice that

\[ f(1) = 2 - 1 = 1 > 0, \quad f(2) = 2^2 - 2^3 = 4 - 8 = -4 < 0 \]

Thus \((1, 2)\) is an interval of length 1 containing a root (solution) of the equation
Problem 5. Given the function

\[ f(x) = x^2 - \sqrt{x} + 2 \]

- find the value of the derivative \( f'(1) \)
- write the equation of the tangent line at \( a = 1 \).

Solution Notice that \( \sqrt{x} = x^{\frac{1}{2}} \). Use the power rule to obtain the derivative function

\[ f'(x) = 2x - \frac{1}{2\sqrt{x}} \]

Thus, at \( x = 1 \),

\[ f'(1) = 2 - \frac{1}{2} = \frac{3}{2} \]

The equation of the tangent line at \( a = 1 \) is

\[ y = f(1) + f'(1) \cdot (x - 1) \]

Notice that \( f(1) = 1 - 1 + 2 = 2 \).

Therefore, the equation of the tangent line at \( a = 1 \) is

\[ y = 2 + \frac{3}{2} \cdot (x - 1) \]

which may be also written as

\[ y = \frac{3}{2}x + \frac{1}{2} \]
**Problem 6.** Given the function

\[ f(x) = \begin{cases} 
  x + 2, & \text{for } x < 1 \\
  ax^2 + b, & \text{for } x \geq 1 
\end{cases} \]

find the constants \( a \) and \( b \) such that the function is continuous and differentiable at \( x = 1 \).

**Solution** Notice that

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x + 2) = 3, \quad f(1) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (ax^2 + b) = a + b
\]

Therefore, the function is *continuous* at \( x = 1 \) if we require that

\[ a + b = 3 \]

Notice that the derivative at \( x = 1 \) is defined as

\[ f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} \]

We have

\[ f'(1-) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{x + 2 - (a + b)}{x - 1} \quad \text{use } a+b=3 \quad \lim_{x \to 1^-} \frac{x - 1}{x - 1} = 1 \]

and

\[ f'(1+) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} \]

\[ = \lim_{x \to 1^+} \frac{ax^2 + b - (a + b)}{x - 1} = \lim_{x \to 1^+} \frac{ax^2 - 1}{x - 1} = \lim_{x \to 1^+} \frac{a(x - 1)(x + 1)}{x - 1} = 2a \]

For the function to be *differentiable* at \( x = 1 \) we require that \( f'(1-) = f'(1+) \),

\[ 2a = 1 \quad \text{thus} \quad a = \frac{1}{2} \]

Then, since \( a + b = 3 \) we get the value of \( b \) as

\[ b = 3 - \frac{1}{2} = \frac{5}{2} \]