Circulant Graphs, J(v,k,i) Graphs, and Homomorphisms

Two nice families and a novel look at graph colorings
Circulant Graphs

How to make nice graphs: (a first try)...

1. Start with $\mathbb{Z}_n$. (verts)
2. Choose $C \subseteq \mathbb{Z}_n$.
3. Put $a \sim b$ iff $a - b \in C$.

Call the graph $X(\mathbb{Z}_n,C)$. Observe:
   -- Need $C$ inverse closed: else (3) isn’t “iff”.
   (or $X$ is directed)
   -- Need $C$ not to have 0: else we get loops.

**Def.** If $C \subseteq \mathbb{Z}_n \setminus \{0\}$ is inverse closed, then $X(\mathbb{Z}_n,C)$ is called a **circulant graph**.

Observe:
   Send $x$ to $-x$: automorphism!
   Send $x$ to $x+k$: automorphism!

Consequence:
   $D_n$ is subgroup of $\text{Aut}(X)$.

... when is $X(\mathbb{Z}_n,C)$ connected?

The graph $X(\mathbb{Z}_8,C)$ where $C = \{1, 7, 2, 6\}$
Def. Suppose X,Y are graphs. A function f: V(X)→V(Y) is a **homomorphism** whenever

\[ u \sim v \text{ in } X \quad \text{implies} \quad f(u) \sim f(v) \text{ in } Y. \]

**Notice:**

1. No assumption f is 1-1 or onto (clearly).
2. This is NOT an “if and only if”.
3. If \( u \sim v \), then \( f(u) \neq f(v) \).

- This means \( f(u) \) and \( f(v) \) can be adjacent, even if \( u,v \) are not.
- “Edges must land on edges.”
Example. The following is a homomorphism from $C_8$ to the ‘House’ graph $H$ shown below:
Example. If $X$ is any bipartite graph, there exists a homomorphism from $X$ to $K_2$:  

\[ \begin{array}{c|cccccccc} 
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline 
\end{array} \]
Graph Colorings

- A **k-coloring** of a graph $X$ is a function $f: V(X) \rightarrow \{1, 2, 3, \ldots, k\}$.

- We say $f$ is **proper** whenever:
  
  $$u \sim v \quad \text{implies} \quad f(u) \neq f(v).$$

- The **chromatic number** $\chi$ denotes the minimum $k$ such that there exists a proper $k$-coloring.

Quick Check: Find $\chi$ for these:
Graph Colorings

Does there exist a homomorphism $P \to C_4$?
**Easy Fact:**

“Proper k-colorings” are equivalent to “homomorphisms to $K_k$.”

**Why:**

Send vertices colored “$j$” to vertex “$j$” in $K_k$.

- Color classes are independent sets, so they can be sent to one vertex. All edges go between color classes, so they will safely land on edges in $K_k$.
- Conversely, pre-images of vertices in $K_k$ are independent, so they can share a color.
Homomorphisms $X \rightarrow X$

A few things to notice:

- An isomorphism is \textit{not} merely a bijective homomorphism.
- Compositions of homomorphisms are homomorphisms.

\textbf{Def.} A homomorphism from $X$ to $X$ is called an \textit{endomorphism}.

- The set of all these is NOT a group (inverses?).
- It’s a \textit{monoid} [set w/assoc binary operation and identity].

Fun with a monoid!

- find all endomorphisms
- name them
- make an operation table
- have fun!
**Def.** If an endomorphism of $X$ is the identity function on its image, we call it a **retraction**. We call the subgraph induced on the image a **retract** of $X$.

Is the square a retract of the cube?

... is $C_5$ a retract of the Petersen graph?
How to make nice graphs: (a different trick)…

1. Choose integers \( v \geq k \geq i \geq 0 \).
2. Start with all \( k \)-sets in a \( v \)-set
3. Put \( a \sim b \) iff \( a,b \) intersect in \( i \) elts.

Call the graph \( J(v,k,i) \). At first glance, this graph looks kind of ugly.

But… note that any permutation of the set \( \{1, 2, 3, 4, 5\} \) is an automorphism.

So this graph has at least \( 5! = 120 \) automorphisms!

Lem. 1.6.2 says \( \text{Aut}[J(v,k,i)] \) always has a subgroup isomorphic to \( \text{Sym}(v) \).

…why should they have required \( k \neq 0,v \)?
Combinatorial Graphs $J(v,k,i)$

Note that the complement of $J(5,2,1)$ is actually $J(5,2,0)$. (why?)

Rather than valency 6, this new graph has valency 3 (cubic).

If we start with the permutation $(1,2,3,4,5)$ then we see that it sends:

13 $\rightarrow$ 24 $\rightarrow$ 35 $\rightarrow$ 14 $\rightarrow$ 25 $\rightarrow$ 13
and
12 $\rightarrow$ 23 $\rightarrow$ 34 $\rightarrow$ 45 $\rightarrow$ 15 $\rightarrow$ 12

which we can use to place the vertices into 2 concentric circles and get a better drawing…

Graph drawing is an important topic!
In fact, $J(5,2,0)$ is the Petersen graph!

When $i=0$, the graphs $J(v,k,i)$ are called the Kneser graphs. When $i=k-1$, they are called the Johnson graphs.

With $J(v,k,i)$ graphs, we typically assume that $v \geq 2k$ because there are some obvious isomorphisms among some of the $J(v,k,i)$. Namely…

**Lem.** $J(v,k,i) \cong J(v, v-k, v-2k+i)$.

**proof.** Take complements of the k-sets. (check that this works as advertised!)

The graph $J(5,2,0)$, and complement of $J(5,2,1)$.
Unlocking more of Ch.1 HW

After working through Sections 1.4-1.6, the following exercises in Ch.1 are likely to be accessible:

# 4, 5, 6, 7, 8, 18

(Recall the plan is to collect 8 problems from each chapter. This chapter has 8 sections and 26 exercises in total.)
Extension #1

Some further thoughts about circulant graphs:

– When is $X(\mathbb{Z}_n, C)$ connected?
– How does $n$ affect the # of options for $C$ such that $X$ is connected?
– Can we generalize beyond circulant graphs by substituting other finite groups $G$ for $\mathbb{Z}_n$ in this construction? How would that work?
Further thoughts about homomorphisms:

– Recall that ‘isomorphism’ does not equal ‘bijective homomorphism’. Why?
– Is an automorphism the same as a bijective endomorphism?
– Explore the endomorphism monoid of some small graph(s). Look at the operation table and look for submonoids, etc.
Further thoughts:

– Is $C_5$ a retract of $P$?
– When is a cycle $C_n$ a retract of a graph?
– When, if ever, is the complement of a Kneser graph a Johnson graph?