Linear Extensions of LYM Posets

Ewan Kummel
Preliminaries

- A binary relation $\leq$ on a set $P$ is defined to be a **partial order** on $P$ when $\leq$ is reflexive, transitive, and antisymmetric.
- We will refer to the pair $(P, \leq)$ as the **partially ordered set**, or **poset**, $P$.
- The relation is a **total order** if $X$ and $Y \in P$ implies that $X \leq Y$ or $Y \leq X$.
- A map $\sigma$ from a poset $P$ to a poset $Q$ is **order preserving** if, for each $X$ and $Y \in P$, $X \leq_P Y$ implies that $\sigma(X) \leq_Q \sigma(Y)$.
- An order preserving bijection $\varepsilon : P \rightarrow Q$ is a **linear extension** of $P$ if $Q$ is totally ordered.
- Two posets are isomorphic if there is an invertible, order preserving, bijection between them.
A binary relation \( \preceq \) on a set \( P \) is defined to be a **partial order** on \( P \) when \( \preceq \) is reflexive, transitive, and antisymmetric.

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We define $e(P)$ to the the size of $E(P)$.

A trivial upper bound is

$$e(P) \leq |P|!$$

(The right hand side counts the number of total orderings of the set $P$.)
Counting The Linear Extensions of a Finite Poset

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Subsets of Posets

Let $Q$ be a subset of a partially ordered set $P$.

- $Q$ is an **order ideal** if for each $X \in Q$, $Y \preceq X$ implies $Y \in Q$ for all $Y \in P$.
- $Q$ is a **filter** if for each $X \in Q$, $X \preceq Y$ implies $Y \in Q$ for all $Y \in P$.
- $Q$ is a **chain** if for each $X$ and $Y \in Q$ either $X \preceq Y$ or $Y \preceq X$.
- $Q$ is an **antichain** if for each $X$ and $Y \in Q$ neither $X \preceq Y$ nor $Y \preceq X$. 
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Linear Extensions, Order Ideals, and Antichains

- If $\varepsilon$ is a linear extension of a poset $P$ then the elements of $P$ can be written $X_1, X_2, \ldots, X_{|P|}$ so that $X_i \preceq \varepsilon X_j$ if and only if $i \leq j$. In fact, this sequence uniquely characterizes $\varepsilon$.

- Letting $O_i = \{X_1, X_2, \ldots, X_i\}$ we can construct a sequence of order ideals $O_1, O_2, \ldots, O_{|P|}$ of $P$. Again, this sequence uniquely characterizes $\varepsilon$.

- Given an ideal $O$ of $P$, we define the map $a$ by

  \[ a(O) = \min \{P - O\} \]

  $a(O)$ is always an antichain, called the choice antichain of $O$. This map establishes a bijection between the order ideals of $P$ and the antichains of $P$.

- This allows us to translate the sequence of ideals $O_1, O_2, \ldots, O_{|P|}$ into a sequence of antichains $a(O_1), a(O_2), \ldots, a(O_{|P|})$. This sequence also uniquely characterizes $\varepsilon$. 
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The Choice Antichain

- Intuitively, the choice antichain of $O$ is the set of every element $X$ of $P - O$ so that the set

$$O \cup \{X\}$$

is also an ideal of $P$.

For the first given linear extension of $B^3$, we have the following sequences:

<table>
<thead>
<tr>
<th>$X_i$</th>
<th>$O_i$</th>
<th>$a(O_i)$</th>
</tr>
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<tbody>
<tr>
<td>$\emptyset$</td>
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<td>${1, 2, 3}$</td>
</tr>
<tr>
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<td>${\emptyset, {1}}$</td>
<td>${2, 3}$</td>
</tr>
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A **rank function** on a poset $P$ is a function $r : P \rightarrow \mathbb{N}$ such that

1. There is a minimal element $X_0 \in P$ so that $r(X_0) = 0$

and

2. $r(X) = r(Y) + 1$ whenever $X$ covers $Y$.

Given any ranked poset $P$,

- the number $\max\{r(X)\}_{X \in P}$ is the **rank** of $P$.
- For any subset $Q$ of $P$, the set $\{X \in Q | r(X) = k\}$ is denoted by $Q_k$.
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The LYM Property

Let $P$ be a rank $n$ poset, with whitney numbers $N_0, N_1, \ldots, N_n$. $P$ has the **LYM property** if for each antichain $A \in P$,

$$\sum_{k=0}^{n} \frac{|A_k|}{N_k} \leq 1.$$
The whitney number $N_k$ of $B^5$ is the binomial coefficient $\binom{5}{k}$.

The antichain $A$ has $|A_0| = |A_4| = |A_4| = 0$, $|A_1| = |A_3| = 1$, and $|A_2| = 3$.

So,

$$\sum_{k=0}^{5} \frac{|A_k|}{\binom{5}{k}} = \frac{1}{5} + \frac{3}{10} + \frac{1}{10} = \frac{3}{5} < 1$$
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The Boolean Lattice

Theorem

(The LYM Inequality) Let $\mathcal{A}$ be an antichain in the Boolean Lattice $\mathcal{B}^n$ and let $\mathcal{A}_k$ be the set of all rank $k$ nodes in $\mathcal{A}$. Then

$$\sum_{k=0}^{n} \frac{|\mathcal{A}_k|}{\binom{n}{k}} \leq 1.$$
\( B^n \) contains exactly \( n! \) maximal chains.

If \( X \in B^n \) and \( r(X) = k \) then \( X \) generates an ideal of rank \( k \) isomorphic to \( B^k \) and a filter of rank \( n - k \) isomorphic to \( B^{n-k} \). It follows that there are exactly \( k!(n-k)! \) maximal chains in \( B^n \) containing \( X \).

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Given any antichain $A$ and any chain $C$ of any poset $P$, $A \cap C$ contains at most 1 element.

Therefore, there are exactly

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Since there are at most $n!$ maximal chains in $B^n$ containing some member of $\mathcal{A}$,

$$\sum_{k=0}^{n} |\mathcal{A}_k| \frac{k!(n-k)!}{n!} \leq 1.$$

Dividing through by $n!$ gives

$$\sum_{k=0}^{n} \frac{|\mathcal{A}_k|}{\binom{n}{k}} \leq 1.$$
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Dividing through by \(n!\) gives

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\]

□
Probabilistic Arguments

We will be using a discrete probability distribution over $E(P)$ to get an upper bound on its size, $e(P)$.

- A function $\rho$ from a finite set $E$ to the interval $[0, 1]$ is a probability distribution over $E$ if

$$\sum_{x \in E} \rho(x) = 1.$$

- A weight function on $P$ is a function $w : \mathcal{P}[P] \to \mathbb{R}^+$ so that for every subset $Q$ of $P$,

$$w(Q) = \sum_{X \in Q} w(X).$$

For each antichain $A$ of $P$, the function $\rho_A : A \to \mathbb{R}$ defined by

$$\rho_A(X) = \frac{w(X)}{w(A)}$$

is a probability distribution over $A$.  

Probabilistic Arguments

We will be using a discrete probability distribution over $E(P)$ to get an upper bound on its size, $e(P)$.

- A function $\rho$ from a finite set $E$ to the interval $[0, 1]$ is a **probability distribution** over $E$ if

$$\sum_{x \in E} \rho(x) = 1.$$ 

- A **weight function** on $P$ is a function $w : \mathcal{P}[P] \to \mathbb{R}^+$ so that for every subset $Q$ of $P$,

$$w(Q) = \sum_{X \in Q} w(X).$$

For each antichain $A$ of $P$, the function $\rho_A : A \to \mathbb{R}$ defined by

$$\rho_A(X) = \frac{w(X)}{w(A)}$$

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The Generalized Sha/Kleitman Bound

**Theorem**

Let $P$ be a ranked poset and let $w$ be a weight function on $P$. If $w(A) \leq 1$ for each antichain $A$ of $P$ then

$$e(P) \leq \frac{1}{\prod_{X \in P} w(X)}.$$
Define a procedure for generating linear extensions of $P$ as follows:

\[
O_0 = \emptyset \\
O_{i+1} = O_i + \{X_i\}
\]

where $X_i$ is chosen from $\alpha(O_i)$ with probability $\rho_{O_i}(X_i)$.

The process terminates after the $|P|$th step when $O_{|P|} = P$ and $\alpha(O_{|P|}) = \emptyset$. The generated sequence $O_1, O_2, \ldots, O_{|P|}$ determines a unique linear extension of $P$.

Alternately, given any sequence $O_1, O_2, \ldots, O_{|P|}$, characterizing a linear extension, the construction results in $O_1, O_2, \ldots, O_{|P|}$ only if the choice of $X_i$ at the $i$th stage is exactly the single element of $O_{i+1} - O_i$. 

**Brightwell’s Proof**
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For each partial sequence $O_1, O_2, \ldots, O_{i-1}$, the value $\rho_{O_i}(X_i)$ is exactly the probability that $X_i$ is chosen at the $i$th stage of our construction given that $O_1, O_2, \ldots, O_{i-1}$ have already been constructed.

It follows that, for any linear extension $\varepsilon$ of $P$, the probability that our construction produces $\varepsilon$ is exactly

$$\mu(\varepsilon) = \prod_{i=1}^{|P|} \rho_{O_i}(X_i).$$

where the sequences $X_1, \ldots, X_{|P|}$ and $O_1, O_2, \ldots, O_{|P|}$ are defined as above. Therefore, $\mu$ is a probability distribution over the set $E(P)$ assigning non-zero probability to each element $\varepsilon \in E(P)$. 
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Brightwell’s Proof

By our assumptions, for any order ideal \( O \) and any \( X \in O \), we have

\[
\rho_O(X) = \frac{w(X)}{w(a(O))} \geq w(X).
\]

Since every element of \( P \) appears exactly once in the sequence \( X_1, \ldots, X_{|P|} \),

\[
\prod_{X \in P} w(X) \leq \prod_{i=1}^{|P|} \rho_{O_i}(X_i) = \mu(\varepsilon).
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$$\prod_{X \in P} w(X) \leq \prod_{i=1}^{|P|} \rho_{O_i}(X_i) = \mu(\varepsilon).$$
Finally, since

\[ \sum_{\varepsilon \in E(P)} \mu(\varepsilon) = 1 \]

it follows that

\[ e(P) \cdot \left( \prod_{X \in P} w(X) \right) = \sum_{\varepsilon \in E(P)} \left( \prod_{X \in P} w(X) \right) \leq \sum_{\varepsilon \in E(P)} \mu(\varepsilon) = 1. \]
Brightwell’s Proof

**Corollary**

*If* $P$ *is an LYM poset with whitney numbers* $N_0, N_1, N_2, \ldots, N_n$ *then*

$$e(P) \leq \prod_{i=0}^{n} N_i^{N_i}.$$
Let \( w(X) = \frac{1}{N_r(X)} \), where \( r \) is the rank function on \( P \). Note that \( w \) is a weight function on \( P \).

If \( P \) is LYM, we have \( w(A) \leq 1 \) for every antichain \( A \) in \( P \).

Therefore, by the previous theorem,

\[
e(P) \leq \prod_{X \in P} \frac{1}{w(X)} = \prod_{X \in P} \frac{1}{N_r(X)} = \prod_{X \in P} N_r(X).
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Conclusion

- This bound is achieved by chains, but it is easy to see that it is not attained by any other poset.
- It is not asymptotic but for small values of $n$ it is the best upper bound we have for $\mathcal{B}^n$.

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<th>$\prod_{i=0}^{n} \binom{n}{i}!$</th>
<th>$e(\mathcal{B}^n)$</th>
<th>$\prod_{i=0}^{n} \binom{n}{i}(n)^i$</th>
<th>$2^n!$</th>
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<td>1</td>
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<td>8</td>
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<td>?</td>
<td>$2.78 \times 10^{420}$</td>
<td>$8.58 \times 10^{506}$</td>
</tr>
</tbody>
</table>
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- Using a very sophisticated probabilistic approach Brightwell and Tetali have published an asymptotic bound on $e(B^n)$ given by

$$e(B^n) \leq e^{6 \cdot 2^n \cdot \frac{\ln n}{n}} \prod_{i=0}^{n} \left( \frac{n}{i} \right)!$$

- It first outdoes the Sha/Kleitman bound at $n = 18$ where

$$\prod_{i=0}^{n} \binom{n}{i} \approx 2.10 \times 10^{1173310}$$

and

$$e^{6 \cdot 2^n \cdot \frac{\ln n}{n}} \prod_{i=0}^{n} \left( \frac{n}{i} \right)! \approx 1.58 \times 10^{1169187}.$$
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